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Projective completion of moduli of t-connections on curves in positive and mixed characteristic

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ABSTRACT

We generalize a compactification technique due to C. Simpson in the context of $\mathbb{G}_m$-actions over the ground field of complex numbers, to the case of a universally Japanese base ring. We complement this generalized compactification technique so that it can sometimes yield projectivity results for these compactifications. We apply these projectivity results to the Hodge, de Rham and Dolbeault moduli spaces for curves, with special regards to ground fields of positive characteristic.

Contents

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1. Introduction

C. Simpson has introduced a compactification technique ([40, §11]) which he then applied to compactify the moduli space of flat connections over a curve defined over the complex numbers. This paper has grown from the need in [8] to generalize this compactification technique so that it leads to compactifications of the moduli spaces that appear in the Non Abelian Hodge Theory of a curve defined over an algebraically closed field of arbitrary characteristic, or over a discrete valuation ring, possibly of mixed characteristic. For more details on these moduli spaces M_{Hod} of t-connections, M_{dR} of connections, and M_{Dol} of Higgs bundles, see §2.2.

Let us discuss a little Simpson's compactification technique leading to the compactification of the moduli space M_{dR} of connections on a curve C over the complex numbers. First, he constructs the moduli space of t-connections as a $\mathbb{G}_m$-equivariant morphism $\tau : M_{Hod} \to \mathbb{A}^1$, where a t-connection on C is sent to the scalar value t. The action is: multiply a t-connection by a non-zero scalar. For $t = 0$, we have the Dolbeault moduli space M_{Dol} of Higgs bundles, and for $t = 1$ we have the de Rham moduli space of connections. There is the Hitchin morphism $h_{Dol} : M_{Dol} \to A$ (A a suitable affine space parametrizing spectral curves for C in the cotangent bundle of C). The fiber N_{Dol} over the point $o \in A$ corresponding to the spectral curve rC –rank times the zero section– is compact and is also the set of points in M_{Hod} that admit infinity limits for the $\mathbb{G}_m$-action. Simpson sets $\overline{M_{dR}} := (M_{Hod} \setminus N_{Dol})/\mathbb{G}_m$; this way the boundary $\overline{M_{dR}} \setminus M_{dR} = (M_{Dol} \setminus N_{Dol})/\mathbb{G}_m$. This definition is simple-minded. On the other hand, the proof that the quotient exists as a separated proper scheme over $\mathbb{C}$ is quite clever and intricate (it does not use the methods from D. Mumford's Geometric Invariant Theory). Simpson proves a more general result ([40, §11]), where one takes the quotient by $\mathbb{G}_m$ of a suitable $\mathbb{G}_m$-variety U/S over a complex variety S endowed with the trivial $\mathbb{G}_m$-action. This is what we mean by Simpson's compactification technique. The application to the compactification of M_{dR} is a special case; see diagram (49) in the proof of Theorem 2.14.

The first set of results of this paper are stated in §2.1 and are proved in the lengthy and technical §3. We generalize Simpson's compactification technique in Theorems 2.6 and 2.7. In short, the set-up $U/S/\mathbb{C}$ above, is replaced by one of the form $U/S/B/J$, where B is a base scheme over a universally Japanese ring J, and the multiplicative group acting is $\mathbb{G}_{m,B}$. This level of generality seems to be the natural one in view of A. Langer's results yielding, as special cases of his [27, Tm. 1.1], the moduli spaces we work with for families of curves over a base defined over such a ring. This covers the case of discrete valuation rings, which is of interest in [8]. We complement these results with the compactification and projectivity criteria in Theorem 2.8; here one works with an equivariant morphism $U/S \to U'/S$, and this is useful in our applications, as the moduli spaces we work with do carry such morphisms, such as the Hitchin morphism h_{Dol} seen above, and we want to compactify domain and target, while keeping track of the morphism.

The second set of results are compactification results for the moduli spaces M_{Hod}, M_{dR} and M_{Dol}. Recall (§2.2) that we have natural morphisms exiting these moduli spaces: the proper Hitchin morphism h_{Dol}; the structural morphism $\tau_{Hod} : M_{Hod} \to \mathbb{A}^1$. In positive characteristic, we also have: the Hodge-Hitchin morphism $h_{Hod} : M_{Hod} \to A' \times \mathbb{A}^1$ (here A' is a suitable affine space parameterizing the spectral curves for the Frobenius twist of the curve); the de Rham-Hitchin morphism $h_{dR} : M_{dR} \to A'$. Our compactification results for these moduli spaces and the associated morphisms are stated in §2.2 and proved in §4, as an application of Theorem 2.8: Theorem 2.13 (M_{Hod}); Theorem 2.14 (positive characteristic, M_{Hod} and h_{Hod}); Theorem 2.17 (M_{dR}); Theorem 2.18 (positive characteristic, M_{dR} and h_{dR}); Theorem 2.19 (M_{Dol}); Theorem 2.18 (positive characteristic, M_{Dol} in relation to M_{Hod}).

In fact, we also prove projectivity results concerning the aforelisted natural morphisms exiting these moduli spaces. We prove, using the known fact that the Hitchin morphism h_{Dol} is proper, that the morphisms $\overline{\tau_{Hod}} : \overline{M_{Hod}} \to \mathbb{A}^1$, h_{Hod} and h_{dR} are proper, in fact projective.

The properness of h_{dR} has been proved by M. Groechenig [18], who deduces it from the properness of the Hitchin morphism. The Hodge-Hitchin morphism h_{Hod} has been introduced by Y. Laszlo and C. Pauly, who proved ([29, Pr. 5.1]) that it is proper when restricted over $o_{A'} \times \mathbb{A}^1$, where $o_{A'}$ is the "origin" of A' (t-connections with nilpotent p-curvature). A. Langer's [27, statement at the top of p. 531 and Th. 5.1] implies that h_{Hod} is proper; see Remark 2.16. In either case, one applies a variant of the Langton technique to a related Hitchin morphism, and deduces from it the desired conclusion. The proof we offer is via the compactification theorems we prove, but also relies on Langer's Langton-type result [27, Th. 5.1].

The purpose of the Appendix §5 is stated in §5.1: in short, one wants to extend, under favorable circumstances, the techniques and results in [7] concerning specialization morphisms in cohomology, from a situation over the complex numbers, to the one over a discrete valuation base ring. This entails making sure that: we have suitable compactifications (this is achieved by the compactifications in §2.2); we have the correct formalism of perverse sheaves for schemes over a discrete valuation ring (this is confirmed in §5.2); we carefully revisit [7, §4] and make sure that some potential issues due to positive or mixed characteristic are ironed out, at least under favorable circumstances (this is done in the technical §5.3).

The results of this Appendix §5, which relies heavily on the results in §2.2, are used in [8].

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2. Statement of the main results

2.1. Compactification and projectivity results

In order to prove the compactification Theorems in §2.2 concerning Hodge, de Rham and Dolbeault moduli spaces associated with curves over some suitable base schemes, we first need to prove the Projectivity Theorem III 2.8. In turn, to prove this latter result, we need to prove the Compactification Theorem II 2.7, which is a direct consequence of the Compactification Theorem I 2.6, the proof of which takes the bulk of this paper.

The Compactification Theorems I 2.6 and II 2.7 together generalizes Simpson's Compactification technique [40, Thm. 11.1, 11.2], which is stated and proved by C. Simpson over the field of complex numbers, to the case over a base scheme as in Assumption 2.1.

The Projectivity Theorem III 2.8 is the arbitrary characteristic counterpart to [6, Prop. 3.2.2]. In fact, by using the notation of Theorem 2.8, this same theorem replaces the assumption that $Z \to Z'$ is proper, with the weaker assumption that U is the preimage of U'. This improvement, coupled with auxiliary properness results, affords proofs of properness and of projectivity of certain morphisms and objects arising in Non Abelian Hodge Theory; see §2.2 and their proofs in §4.

The Compactification Theorem I 2.6 can also be viewed as a partial generalization (replace the ground field with the base variety S) of the main theorem in [5, p.11] to the relative case. For a comparison between Theorem 2.6 and the main theorem of [5, p.11]), see Remark 2.10.

Let us introduce the setup for the Compactification Theorem I 2.6. This setup is similar to the one in [40, p. 44]; the main difference is that we work over a base scheme B over a universally Japanese ring J [41, 032E]), while Simpson works over the complex numbers (i.e. $J = \mathbb{C}$). (Added in revision: the paper [28] allows to merely assume B to be Noetherian and drop J.)

Assumption 2.1 ***(Setup for the Compactification Theorem I 2.6).*** The following assumptions concerning schemes $X/S/B/J$ remain in vigour up to and including Theorem 2.6. Let J be a universally Japanese ring. Let B and S be noetherian schemes. Let $S \to B \to J$ be separated morphisms of finite type. Assume that S admits an invertible sheaf that is ample, and an invertible sheaf that is ample relative to B; this ensures that there is a B-morphism that is a locally closed embedding of S into $\mathbb{P}^N_B$ for some $N > 0$. Let $X \to S$ be a projective morphism. Let $\mathbb{G}_{m,B} := \mathbb{G}_m \times_{\mathbb{Z}} B$. Let $\mu : \mathbb{G}_{m,B} \times_B X \to X$ be a $\mathbb{G}_{m,B}$-action on X covering the trivial $\mathbb{G}_{m,B}$-action on S. Assume that X admits a $\mathbb{G}_{m,B}$-linearized ample line bundle.

Our next goal is to state Theorem 2.6 and, to this end, we need some preparation.

Limit points, fixed points. Let us recall the definition of limits of a point in X under the $\mathbb{G}_{m,B}$-action μ. Let T be a B-scheme. Let $x \in X(T)$. Let μ_x be the orbit morphism defined by the following compositum:

$$\mu_x : \mathbb{G}_{m,T} := \mathbb{G}_{m,B} \times_B T \xrightarrow{id \times x} \mathbb{G}_{m,B} \times_B X \xrightarrow{\mu} X. \tag{1}$$

If μ_x extends to a morphism $\widetilde{\mu_x} : \mathbb{A}^1_T \to X$, then this extension is unique. In this case, we say that $\lim_{t\to 0} t\cdot x$ exists and we set it to be the restriction of $\widetilde{\mu_x}$ to $0_T \subset \mathbb{A}^1_T$. Clearly, $\lim_{t\to 0} t\cdot x$ is an T-point of X. Similarly, if μ_x extends to a morphism $\widetilde{\mu_x}' : \mathbb{P}^1_T \setminus 0_T \to X_T$, then this extension is unique, and we set $\lim_{t\to\infty} t \cdot x$ to be the restriction of $\widetilde{\mu_x}'$ to $\infty_T \subset \mathbb{P}^1_T \setminus 0_T$, which is also an T-point of X. By [10, XII Cor. 9.8], there exists the closed subscheme $V \subset X$ of fixed points for the $\mathbb{G}_{m,B}$ action.

A partial order. Let us introduce a partial order on the Zariski points of V as in [40, p. 44] via the following definition. The weight argument as in [40, p. 44] shows that the upcoming relation $\leq$ indeed defines a partial order on the Zariski points of V.

Definition 2.2 ***(The Partial Order*** $\leq$ ***on the Zariski points of*** V*).* Let u and v be two Zariski points of V. Define a relation $\leq$ as follows: $u \leq v$ if there exists a finite sequence of Zariski points $x_1, ..., x_m$ of X such that $\lim_{t\to 0} t \cdot x_1 = u$, $\forall 1 \leq l \leq m-1$, $\lim_{t\to\infty} t \cdot x_l = \lim_{t\to 0} x_{l+1}$, and $\lim_{t\to\infty} t \cdot x_m = v$.

If $u \leq v$, then we say that u is more zero than v, and v is more infinity then u.

Definition 2.3 ***(Partitions*** $V = V^+ \cup V^-$*).* We consider partitions of V with the following properties. Let V_+ and V_- be two disjoint closed and open subschemes of V with the property that $V = V_+ \cup V_-$. In addition, we require the following: if u is more zero than a point in V_+, then $u \in V_+$; if v is more infinity than a point in V_-, then $v \in V_-$.

Concentrators. Let Z be any $\mathbb{G}_m$-stable closed subscheme of X. Let the 0-concentrator functor Φ_0 be the subfunctor of X such that for any B-scheme T, a T-point x of X is in $\Phi_0(T)$ if and only if $\lim_{t\to 0} t \cdot x$ exists and lies in $Z(T)$. By [22, §4.5], we see that Φ_0 is represented by a scheme $X_0(Z)$ with a morphism $X_0(Z) \to X$ that is locally over $X_0(Z)$ a locally closed immersion (note that $X_0(Z) \to X$ may not be a locally closed immersion, see [22, §4.6]). Similarly, we can define the ∞-concentrator Φ_∞ and the scheme morphism $X_\infty(Z) \to X$.

Definition 2.4 *(**Set Theoretic Partition** $X = Y_+ \cup Y_- \cup U$).* We fix a partition $V = V^+ \cup V^-$ of the fixed point set as in Definition 2.3. We define Y_+ to be the set theoretic image of $X_\infty(V_+) \to X$, and define Y_- to be the set theoretic image of $X_0(V_-) \to X$. We define the set $U := X \setminus (Y_+ \cup Y_-)$.

Remark 2.5. Let us show that the sets $Y_\pm$ in Definition 2.4 are disjoint. If there were $x \in Y_+ \cap Y_-$, then x would be more zero than a point $u \in V_+$, and more infinity than a point $v \in V_-$. We would then have that u is more zero than $v \in V_-$. By Definition 2.3, we would have that $u \in V_+ \cap V_-$, contradicting that $V_+ \cap V_- = \emptyset$.

Recall that a uniform (resp. universal) geometric quotient $A \to B$ is a geometric quotient whose formation commutes with flat (resp. arbitrary) base change $B' \to B$.

Theorem 2.6 *(**Compactification Theorem I**). Assumption 2.1 on $X/S/B/J$ are in vigour. Fix a partition of the fixed point set $V = V^+ \cup V^-$ as in Definition 2.3 and let $X = Y_+ \cup Y_- \cup U$ be the corresponding set theoretic partition as in Definition 2.4. We have that*

(1) Both Y_+ and Y_- are closed inside X.
(2) The uniform geometric quotient $U \to U/\mathbb{G}_{m,B}$ exists, with $U/\mathbb{G}_{m,B}$ an S-scheme.
(3) The morphism $U/\mathbb{G}_{m,B} \to S$ is universally closed.
(4) The morphism $U/\mathbb{G}_{m,B} \to S$ is separated, thus, in view of (3) above, proper.

Proof. The proof occupies the whole of §3. □

Theorem 2.7 *(**Compactification Theorem II**). Assumption 2.1 on $S/B/J$ are in vigour. Suppose Z/S is an S-scheme with a $\mathbb{G}_{m,B}$-action that is compatible with the trivial $\mathbb{G}_{m,B}$-action on S. Assume that there is a $\mathbb{G}_{m,B}$-linearized relatively ample line bundle on Z/S. Suppose that the fixed point set $W \subseteq Z$ is proper over S, and that for any $z \in Z$ the limit $\lim_{t\to 0} t \cdot z$ exists in W. Let $U \subseteq Z$ be the subset of points z such that the limit $\lim_{t\to\infty} t \cdot z$ does not exist in Z. Then U is open and there exists a uniform geometric quotient $U \to U/\mathbb{G}_{m,B}$ by the action of $\mathbb{G}_{m,B}$. This geometric quotient is separated and proper over S.*

Proof. This follows from Theorem 2.6 in the same way in which [40, Thm. 11.2] follows from [40, Thm. 11.1]. We only reproduce some of the highlights of the proof. Use the $\mathbb{G}_{m,B}$-linearized relatively ample line bundle on Z/S to embed Z/S $\mathbb{G}_{m,B}$-equivariantly into some $\mathbb{P}^N_S$ as a locally closed subvariety. Take the closure and call it X/S. Let $V \subseteq X$ be the fixed point set. Define $V_+ := W$ to be the fixed point set in Z. Let $V_- := V \cap (X \setminus Z)$. The rest of the proof consists of showing that V_+ and V_- have the desired properties, and that U, as it is defined in the statement of this theorem, is indeed $X \setminus (Y_+ \cup Y_-)$. At this juncture, one applies Theorem 2.6. □

Setup for the Projectivity Theorem III 2.8. Assumption 2.1 on $S/B/J$ are in vigour. Let Z and Z' be varieties over S, endowed with a $\mathbb{G}_{m,B}$-action covering the trivial $\mathbb{G}_{m,B}$-action over S, so that the structural morphisms $Z, Z' \to S$ are $\mathbb{G}_{m,B}$-equivariant. Let $Z \to Z'$ be a $\mathbb{G}_{m,B}$-equivariant S-morphism.

Theorem 2.8 *(**Projectivity Theorem III**). Let $U \subseteq Z$ ($U' \subseteq Z'$, resp.) be the subset such that the ∞-limits do not exist. Assume that*

(a) Z/S and Z'/S carry relatively ample line bundles admitting $\mathbb{G}_{m,B}$-linearizations.
(b) The fixed point set $V \subseteq Z$ is proper over S.
(c) The 0-limits exist in Z.
(d) At least one of the following two conditions is met

(i) the $\mathbb{G}_{m,B}$-equivariant S-morphism $Z \to Z'$ is surjective;
(ii) the fixed point set $V' \subseteq Z'$ is proper over S and the 0-limits exist in Z'.

(e) U is the preimage of U' (this is automatic if $Z \to Z'$ is proper).

Then:

(1) U (U', resp.) is open in Z (Z', resp.);
(2) The morphism $U \to U'$ descends to a proper S-morphism $U/\mathbb{G}_{m,B} \to U'/\mathbb{G}_{m,B}$ between the geometric quotients, both of which are proper and separated over S;
(3) (a) the descended morphism $U/\mathbb{G}_{m,B} \to U'/\mathbb{G}_{m,B}$ is projective;
(b) if, in addition, $(U'/\mathbb{G}_{m,B})/S$ is also projective, then $(U/\mathbb{G}_{m,B})/S$ is projective.

Proof. The proof is identical to the one in [6, Prop. 3.2.2]. Note that in [6] the current assumption (e) is replaced by the assumption that Z/Z' is proper; the proof in [6] works with the current assumption in place of the properness assumption on Z/Z'.

For the reader's convenience, we discuss briefly the structure of the proof. Parts (1,2) can be proved along the same lines of the proof of Theorem 2.7. We simply note the following: the assumption (d.i.) on surjectivity implies easily the assumption (d.ii) on the properness of the fixed point set and the existence of 0-limits. One applies the Compactification Theorem II 2.7 to Z and to Z' to find the uniform geometric quotients $U/\mathbb{G}_{m,B}$ and $U'/\mathbb{G}_{m,B}$. The descended morphism $U/\mathbb{G}_{m,B} \to U'/\mathbb{G}_{m,B}$ between the uniform geometric quotients arises from the $\mathbb{G}_{m,B}$-equivariance of the morphism $U \to U'$. The properness and separateness over S of these quotients follow from Theorem 2.7. The properness of the descended morphism follows from the properness of $(U/\mathbb{G}_{m,B})/S$.

What needs proof is part (3). Part (3) is proved in [6, Prop. 3.2.2] (the set-up there is the one of characteristic zero, but the proof works for arbitrary base scheme B).

The key part is (3.a):

The proof in [6, Prop. 3.2.2] relies on Kempf's Descent Lemma [13, Thm. 2.3], which is stated over fields of characteristic zero. A generalization of Kempf's Descent Lemma to the case over a more general scheme can be found in [1, Thm. 10.3] and [36, Thm. 1.3.(iii)].

To apply [1, Thm. 10.3], we need to show that: (i) The uniform geometric quotient $U/\mathbb{G}_{m,B}$ given by Theorem 2.6 is a good and tame moduli space for the quotient stack $[U/\mathbb{G}_{m,B}]$; (ii) some tensor power of the $\mathbb{G}_{m,B}$-linearized ample line bundle on U has trivial stabilizer action at closed points, in the sense of [1, Def. 10.1].

(i) follows from Remark 3.9. (ii) follows from the fact that the stabilizers of the closed points of U are finite subgroup schemes of $\mathbb{G}_m$ over the corresponding residue fields, and that for a finite group scheme G of order n over a field, the n-th power morphism $g \mapsto g^n : G \to G$ is the identity morphism, see [32, Prop. 11.32].

Then one proves the $(U/\mathbb{G}_{m,B})/(U'/\mathbb{G}_{m,B})$-ampleness of the descended line bundle by observing that it is ample on the fibers of the proper morphism $(U/\mathbb{G}_{m,B})/(U'/\mathbb{G}_{m,B})$. □

Remark 2.9 ***(Comparison of Theorems 2.6 and 2.7 with [40]).*** The Compactification Theorems I,II 2.6, 2.7 are stated and proved in [40, Thm. 11.1, 11.2] over the complex numbers, where [40] Thm. 11.2 is a corollary to [40] Thm. 11.1, the same way the Compactification Theorem II 2.7 follows from the Compactification Theorem I 2.6.

As it is observed in [6, §3.2], C. Simpson's [40, Thm. 11.1, 11.2] are missing a seemingly necessary hypothesis on the existence of a $\mathbb{G}_m$-linearized X/S-ample line bundle. This minor point out of the way, all the necessary ideas are clearly stated by C. Simpson in [40, §11]. We felt that some details were present only in implicit form, and then only within a characteristic zero setup. Since in this paper we need these results also over a base, we felt the need to write a detailed proof of the Compactification Theorem I 2.6. Again, all the ideas in the proof of the Compactification Theorems I, II are due to C. Simpson.

Remark 2.10 ***(Comparison of Theorem 2.6 with [5]).*** When $S = \mathrm{Spec}(k)$, the set U of Theorem 2.6 is called a sectional set in [5, Def. 1.2]; the same paper also considers semi-sectional sets. The most obvious difference between sectional and semi-sectional sets is that, unlike a sectional set, a semi-sectional set may contain some, but not arbitrary, $\mathbb{G}_m$-fixed point. For a semi-sectional set U', it is also proved in [5, Thm. 3.1] that $U'/\mathbb{G}_m$ is a semi-geometric quotient. The difference between a geometric and a semi-geometric quotient is that a point in a semi-geometric quotient may corresponds to multiple orbits. In this paper, we do not consider semi-sectional sets.

The proof in [5, Thm. 3.1] is obtained by first establishing what are the possible configurations of fixed-point sets for actions of $\mathbb{G}_m$ on projective spaces $\mathbb{P}^N_k$. One equivariantly embeds X in some $\mathbb{P}^N_k$ by using the ample $\mathbb{G}_m$-linearized line bundle. This is followed by an inductive analysis, and here we summarize very roughly, of how the fixed-point set on X is related to the fixed-point set of the ambient $\mathbb{P}^N_k$. In the relative case, it is not clear to us how to piece together the possible global configurations of the fixed-point sets of $(\mathbb{P}^N_S, X)$ fiber-by-fiber over S. Therefore, it is not clear to us how to modify the proof in [5, Thm. 3.1] to make it work in the relative case over S we are working with.

Remark 2.11 ***(Comparison of Theorem 2.8 with [6]).*** (a) The items (1), (3), and (4) of Projectivity Theorem III 2.8 are essentially borrowed from [6, Prop. 3.2.2]. [6] is stated and proved in characteristic zero, but, once the Compactification Theorems I, II, 2.6 and 2.7 are in place, the proof carries over to arbitrary characteristic.

(b) Moreover, we remove from [6, Prop. 3.2.2] the hypothesis that $Z \to Z'$ is proper, and we replace it with the weaker hypothesis that U is the preimage of U'. Observe that the preimage of U' sits inside U automatically. If one assumes that $Z \to Z'$ is proper, then one shows that the preimage of U' is U. In all the applications of the Projectivity Theorem III that we provide in this paper, i.e. in §2.2, the sets U and U' are constructed, and the preimage of U' is verified to be U by inspection of the construction. In all such applications, we have that U'/U is proper. We ignore if the assumptions of Theorem 2.8 imply that U/U' must be proper.

2.2. Applications to projectivity in non Abelian Hodge theory

We introduce the setup for our main projectivity results, Theorems 2.14 (Hodge/t-connections), 2.18 (de Rham/flat connections) and 2.20 (Dolbeault/Higgs bundles).

The context is the one of moduli spaces of t-connections on a curve, which is a kind of umbrella covering, in some sense, Higgs bundles and flat connections. The notion of t-connections was introduced and studied by C. Simpson [40] over the complex numbers, and by Y. Lazslo and C. Pauly [29] in positive characteristic.

Smooth Curves. Let B be a noetherian scheme that is finite type over a universally Janpanese ring J. Let $\pi : C \to B$ be a projective and smooth family of geometrically connected curves. We record such a family of curves as

$$C/B/J. \tag{2}$$

Rank r and degree d. We fix the rank r and degree d of the vector bundles underlying Higgs bundles, connections and t-connections. When relevant, in context, we make further assumptions on rank and degree, and sometimes on the characteristic.

The Hodge Moduli Space. A t-connection on C/B is a triple (E, t, ∇_t), where E is vector bundle, $t \in H^0(B, \mathcal{O}_B)$, and $\nabla_t : E \to E \otimes_{\mathcal{O}_B} \omega_{C/B}$ is an $\mathcal{O}_B$-linear morphism of $\mathcal{O}_C$-modules so that for every f, a local section of $\mathcal{O}_E$, and s, a local section of E, we have that $\nabla_t(fs) = tdf \otimes s + f\nabla_t(s)$.

By [27, Thm. 1.1], there exists a quasi projective B-scheme $M_{Hod}(C/B)$, which is the coarse moduli space of slope semistable t-connections of rank r and degree d on C/B.

Remark 2.12. For the notions of universally/uniformly corepresenting, see [27, Thm. 1.1]. The coarse moduli space uniformly corepresents the functor of semistable families;

the stable part is open and universally corepresents the functor of stable families; when rank and degree are coprime, stability equals semistability, and we have universal corepresentability. In particular, in the stable case, taking fibers commutes with taking the coarse moduli space.

Considering t as a section of $\mathbb{A}^1_B$ over B, the assignment $(t, E, \nabla_t) \mapsto t$ defines a natural morphism of B-schemes:

$$\tau_{Hod}(C/B) : M_{Hod}(C/B) \longrightarrow \mathbb{A}^1_B. \tag{3}$$

Frobenius. Let J be of characteristic $p > 0$, with p a prime number. Let $q : T \to B$ be a B-scheme. Let $fr_T : T \to T$ be the absolute Frobenius, i.e. the identity on the topological space, with comorphism $a \mapsto a^p$. Let $T^{(B)} := T \times_{B, fr_B} B$ be the Frobenius twist of T relative to B. We have the following commutative diagram

$$\begin{tikzcd} T \arrow[r, "Fr_T"'] \arrow[rr, bend left, "fr_T"] \arrow[dr, "q"'] & T^{(B)} \arrow[r, "\sigma_T"'] \arrow[d, "q^{(B)}"] & T \arrow[d] \\ & B \arrow[r, "fr_B"'] & B. \end{tikzcd} \tag{4}$$

The Hodge-Hitchin Morphism. Let J be a field of characteristic $p > 0$. Given any t-connection ∇_t on C/B, [29, §3.5] defines the p-curvature $\Psi(\nabla_t)$ of ∇_t, which is an $\mathcal{O}_C$-linear morphism $E \to E \otimes_{\mathcal{O}_B} \omega_{C/B}^{\otimes p}$. Let $A(C/B, \omega_{C/B}^p)$ be the vector bundle associated with the locally free sheaf $\bigoplus_{i=1}^r \pi_* \omega_{X/B}^{\otimes ip}$ (recall that we have fixed rank r and degree d for the Hodge moduli space). Taking the characteristic polynomial of $\Psi(\nabla_t)$ defines a morphism $cp : M_{Hod}(C/B) \to A(C/B, \omega_{C/B}^p)$. Let $A(C^{(B)}/B, \omega_{X^{(B)}/B})$ be the total space of the vector bundle $\bigoplus \pi_*^{(B)} \omega_{X^{(B)}/B}^{\otimes i}$. The Frobenius pull back $Fr_{C/B}^*$ defines a closed immersion $A(C^{(B)}/B, \omega_{X^{(B)}/B}) \hookrightarrow A(C/B, \omega_{C/B}^p)$. [29, Prop. 3.2] shows that there exists natural factorization of $(cp, \tau_{Hod}) : M_{Hod}(C/B) \to A(C/B, \omega_{C/B}^p) \times \mathbb{A}^1_B$ as

$$M_{Hod}(C/B) \xrightarrow{h_{Hod}(C/B)} A(C^{(B)}/B, \omega_{X^{(B)}/B}) \times \mathbb{A}^1_B \xrightarrow{(Fr_{C/B}^*, \mathrm{Id}_{\mathbb{A}^1_B})} A(C/B, \omega_{C/B}^p) \times \mathbb{A}^1_B. \tag{5}$$

The quasi-projective morphism $h_{Hod}(C/B)$ in (5) is called the Hodge-Hitchin morphism. Note that [29, Prop. 3.2] contains a minor inaccuracy, as it declares the target of H to be $A \times_B \mathbb{A}^1$.

The diagram (5) is made of B-schemes endowed with $\mathbb{G}_{m,B}$-actions so that the morphisms are $\mathbb{G}_{m,B}$-equivariant. The action on M_{Hod} is given by $t \cdot (E, \nabla_s) := (E, \nabla_{ts})$. Let $\mathbb{A}'_i$ (resp. $\mathbb{A}^p_i$) be the direct factor of $A(C^{(B)}/B)$ (resp. $A(C/B, \omega_{C/B}^p)$) that is the vector bundle associated to the locally free sheaf $\pi_*^{(B)} \omega_{X^{(B)}/B}^{\otimes i}$ (resp. $\pi_* \omega_{X/B}^{ip}$). The action on

$A(C^{(B)}/B)$ (resp. $A(C/B,\omega^p_{C/B})$) is given by the standard dilation weight ip actions on each direct factor $\mathbb{A}'_i$ (resp. $\mathbb{A}^p_i$). The action on $\mathbb{A}^1_B$ is the usual weight one dilation action.

Dolbeault Moduli Space and Hitchin Morphism. Let $M_{Dol}(C/B)$ be the coarse moduli space of slope semistable Higgs bundles of fixed rank r and degree d on C/B. $M_{Dol}(C/B)$ is quasi-projective, see [27, Thm. 1.1]. Let $A(C/B;\omega_{C/B})$ be the vector bundle associated to the locally free sheaf $\bigoplus_{i=1}^r \pi_*\omega_{C/B}^{\otimes i}$. Let

$$h_{Dol}(C) : M_{Hod}(C/B) \longrightarrow A(C/B,\omega_{C/B}) \tag{6}$$

be the Hitchin morphism that sends a Higgs field to the coefficients of its characteristic polynomial.

If J is a field of positive characteristic, then there exists a natural isomorphism $A(C/B,\omega_{C/B})^{(B)} \cong A(C^{(B)}/B,\omega_{C^{(B)}/B})$ (See Lemma 4.1). Let $h_{Hod}(C/B)_{0_B} : M_{Hod}(C/B)_{0_B} \to A(C^{(B)}/B)$ be the base change of the Hitchin morphism via the closed immersion $0_B \hookrightarrow \mathbb{A}^1_B$. There exists a natural morphism $M_{Dol}(C/B) \to M_{Hod}(C/B)_{0_B}$ that is bijective on geometric points. Lemma 4.2 shows that there exists the following commutative diagram of $\mathbb{G}_{m,B}$-equivariant morphisms

$$\begin{tikzcd} M_{Hod}(C/B)_{0_B} \arrow[r, "h_{Hod,0_B}"] & A(C^{(B)}/B) \arrow[r, "\simeq"] & A(C/B)^{(B)} \\ M_{Dol}(C/B) \arrow[u] \arrow[r, "h_{Dol}"'] & A(C/B). \arrow[ur, "Fr_{A(C/B)/B}"'] & \end{tikzcd} \tag{7}$$

de Rham Moduli Space and de Rham-Hitchin Morphism. A flat connection is a t-connection with $t = 1$. Let $M_{dR}(C)$ be the moduli space of semistable flat connections of fixed rank r and degree d. By [27, Thm. 1.1], the de Rham moduli space $M_{dR}(C)$ is quasi-projective.

When J is a field of positive characteristic, there is the natural morphism $M_{dR}(C) \to M_{Hod}(C) \times_{\mathbb{A}^1_B} 1_B$, which, by Lemma 4.5, is an isomorphism. The restriction $h_{dR}(C)$ of $h_{Dol}(C)$ to $M_{dR}(C)$ is called the de Rham-Hitchin morphism. Lemma 4.5.(2) shows that the restriction $h_{Hod}(C)_{\mathbb{G}_{m,B}} : M_{Hod}(C) \times_{\mathbb{A}^1_B} \mathbb{G}_{m,B} \to A(C^{(B)}) \times_B \mathbb{G}_{m,B}$ admits a $\mathbb{G}_{m,B}$-equivariant trivialization as $h_{dR}(C) \times \mathrm{Id} : M_{dR}(C) \times_B \mathbb{G}_{m,B} \to A(C^{(B)}) \times_B \mathbb{G}_{m,B}$.

Statement of Results.

Our first result is the Projective Completion of $\tau : M_{Hod} \to \mathbb{A}^1$ Theorem 2.13, to the effect that there is a natural $\mathbb{G}_m$-equivariant projective completion $\overline{\tau} : \overline{M_{Hod}} \to \mathbb{A}^1$ of the morphism $\tau : M_{Hod} \to \mathbb{A}^1$. If we further require that the base ring J is a field of positive characteristic, we can also extend the Hodge-Hitchin morphism $h_{Hod} : M_{Hod} \to A(C^{(B)}/B) \times \mathbb{A}^1$ and prove that the Hodge-Hitchin morphism h_{Hod} is proper, in fact projective (Theorem 2.14). To our knowledge, the properness of h_{Hod} has not been addressed before. [29, Prop. 5.1] addresses the special case of nilpotent t-connections, i.e. the properness of h_{Hod} over the locus $0_A \times \mathbb{A}^1$. In our proof, we leverage on this special case; in fact, we only need the properness of the nilpotent cone N_{Dol}, i.e. that of the Hitchin fiber $h_{Dol}^{-1}(0_A)$ over the origin $0_A \in A(C/B)$. The special case of this projectivity result when rank and degree are coprime (which rules out the case of degree zero, for example) is established by an ad hoc method in [8].

Theorem 2.13 *(**Projective Completion of** $\tau : M_{Hod} \to \mathbb{A}^1$). Let the smooth curve $C/B/J$ be as in (2). We have the following commutative $\mathbb{G}_{m,B}$-equivariant diagram*

$$\begin{tikzcd} M_{Hod}(C/B) \arrow[r, hook] \arrow[d, "\tau_{Hod}(C/B)"'] & \overline{M_{Hod}(C/B)} \arrow[d, "\overline{\tau_{Hod}(C/B)}"] \\ \mathbb{A}^1_B \arrow[r, "\simeq"] & \mathbb{A}^1_B, \end{tikzcd} \tag{8}$$

where:

(1) The top horizontal arrow is an open immersion with dense image, dense in every fiber of $\overline{\tau_{Hod}(C/B)}$;
(2) The morphism $\overline{\tau_{Hod}(C/B)}$ is projective.

Theorem 2.14 *(**Projective Completion of the Hodge-Hitchin Morphism**). In the setup in Theorem 2.13, if we further assume that J is a field of characteristic $p > 0$, then we have the following commutative $\mathbb{G}_{m,B}$-equivariant diagram*

$$\begin{tikzcd} M_{Hod}(C/B) \arrow[r, hook] \arrow[d, "h_{Hod}(C/B)"] \arrow[dd, bend right=60, "\tau_{Hod}(C/B)"'] & \overline{M_{Hod}(C/B)} \arrow[d, "\overline{h_{Hod}(C/B)}"'] \arrow[dd, bend left=60, "\overline{\tau_{Hod}(C/B)}"] \\ A(C^{(B)}/B) \times \mathbb{A}^1_B \arrow[r, hook] \arrow[d, "pr"'] & \overline{A(C^{(B)}/B)} \times \mathbb{A}^1_B \arrow[d, "\overline{pr}"] \\ \mathbb{A}^1_B \arrow[r, "\simeq"] & \mathbb{A}^1_B, \end{tikzcd} \tag{9}$$

where:

(1) The top square is Cartesian, the horizontal arrows are open immersions with dense image, dense in every fiber of $\overline{\tau_{Hod}(C/B)}$ and $\overline{h_{Hod}(C/B)}$.
(2) The morphisms $h_{Hod}(C/B), \overline{h_{Hod}(C/B)}$ and $\overline{pr}$ are proper, in fact projective (pr is affine).
(3) $\overline{A(C^{(B)}/B)}$ is the weighted projective space $\mathbb{P}(1, 1 \cdot p, 2p, \dots, rp) = \mathbb{P}(1, 1, 2, \dots, r)$ associated with the $\mathbb{G}_{m,B}$-variety $\mathbb{A}^1 \times \prod_{i=1}^r \mathbb{A}'_i$, where $\mathbb{G}_m$ acts as standard dilations of weight 1 on $\mathbb{A}^1$ and of weight ip on the remaining factors.

The proofs of Theorems 2.13 and 2.14 are postponed to §4.2.

Remark 2.15. For the stated equality of weighted projective spaces, i.e. keep the first 1 and replace ip by i for $i = 1, \dots, r$, see [9, Prop. 1.3] and [12, §1.3, Proposition]. This should not be confused with the fact that, when dealing with weighted projective spaces, we can replace the vector of weights by a positive integer multiple of it.

Remark 2.16. A. Langer's [27, statement at the top of p. 531 and Th. 5.1] implies that h_{Hod} is proper. On the other hand, in order to have a complete proof of [27], one also needs to prove that the morphism from the moduli of semistable bundles with t-connections to the appropriate moduli space of semistable Higgs bundles (with Higgs field then given by the p-curvature of the t-connection), is proper. A. Langer has very kindly provided us with a proof of this fact in a private communication. Added in revision: the paper [28] provides complete details and proves an even stronger statement.

By taking the fiber over $1_B \in \mathbb{A}^1_B$ of (8) and (9), and by observing that the fiber of $\overline{\tau}$ over the same value is Simpson's compactification $\overline{M_{dR}}$, we immediately deduce the following Theorems 2.17 and 2.18:

Theorem 2.17 *(**Projective Completion of** M_{dR}). Let the smooth curve $C/B/J$ be as in (2). There exists a projective B-scheme $\overline{M_{dR}(C/B)}$ and an open immersion of B-schemes $M_{dR}(C/B) \hookrightarrow \overline{M_{dR}(C/B)}$ with dense image.*

Theorem 2.18 *(**Projectivity of the de Rham-Hitchin morphism**). In the setup in Theorem 2.17, if we further require that J is a field of characteristic $p > 0$, then we have the following Cartesian diagram*

$$\begin{tikzcd} M_{dR}(C/B) \arrow[r, hook] \arrow[d, "h_{dR}=h_{Hod,1_{\mathbb{A}^1}}"'] & \overline{M_{dR}(C/B)} \arrow[d, "\overline{h_{Hod,1_{\mathbb{A}^1}}}"] \\ A(C^{(B)}/B) \arrow[r, hook] & \overline{A(C^{(B)}/B)}, \end{tikzcd} \tag{10}$$

where

1. *The horizontal arrows are open embeddings with dense image;*
2. *The morphisms $h_{dR} = h_{Hod,1_{\mathbb{A}^1}}$ and $\overline{h_{Hod,1_{\mathbb{A}^1}}}$ are projective;*
3. *$\overline{A(C^{(B)}/B)}$ is the weighted projective space in Theorem 2.14.(3);*
4. *The compactification $\overline{M_{dR}}$ is projective.*

In the Dolbeault case, we do not know whether the natural $\mathbb{G}_{m,B}$-equivariant morphism $M_{Dol} \to M_{Hod,0_{\mathbb{A}^1}}$ is an isomorphism. On the other-hand, we use the $\mathbb{G}_{m,B}$-action to obtain projective $\mathbb{G}_{m,B}$-equivariant completions of M_{Dol} and of $M_{Hod,0_{\mathbb{A}^1}}$ which are suitably compatible with the natural $\mathbb{G}_{m,B}$-equivariant morphism $M_{Dol} \to M_{Hod,0_{\mathbb{A}^1}}$. Note that we use the subscripts to denote the fibers: for example, the fiber of h_{Hod} over $0_{\mathbb{A}^1}$ is denoted by $M_{Hod,0_{\mathbb{A}^1}}$.

Theorem 2.19 *(**Projective Completion of the Dolbeault Moduli Space**). Let the smooth curve $C/B/J$ be as in (2). We have the following commutative $\mathbb{G}_{m,B}$-equivariant diagram*

$$\begin{tikzcd} A(C/B) \arrow[d] & M_{Dol}(C/B) \arrow[l, "h_{Dol}"'] \arrow[r] \arrow[d] & M_{Hod,0_{\mathbb{A}^1}}(C/B) \arrow[d] \\ \overline{A(C/B)} & \overline{M_{Dol}(C/B)} \arrow[l, "\overline{h_{Dol}}"'] \arrow[r] & \overline{M_{Hod,0_{\mathbb{A}^1}}(C/B)}, \end{tikzcd} \tag{11}$$

where:

(1) *All the B-schemes in the bottom row of (11) are projective;*
(2) *All the vertical arrows are open immersions with dense image;*
(3) *All the horizontal arrows in (11) are projective;*
(4) *The two horizontal arrows in the right half of (11) are bijective on geometric points and, if degree and rank are coprime, then they are isomorphisms.*
(5) *The morphism $\overline{h_{Dol}} : \overline{M_{Dol}} \to \overline{A}$ is naturally isomorphic to the compactification constructed in [6, Thm. 3.1.1, which uses Thms. 3.2.1 and 3.2.2] ([6] works over the complex numbers, but in view of the compactification and projectivity results of this paper, the construction and results hold in arbitrary characteristic as well).*

When the base ring J is a field of positive characteristic, the Hodge-Hitchin morphism exists, and we can slightly improve Theorem 2.19 as follows:

Theorem 2.20 *(**Projectivity of the Hitchin morphism**). In the setup in Theorem 2.19, if we further require that J is a field of characteristic $p > 0$, then we have the following $\mathbb{G}_{m,B}$-equivariant commutative diagram:*

$$\begin{tikzcd} \overline{M_{Dol}(C/B)} \arrow[rrr] \arrow[ddd, "\overline{h_{Dol}}"'] & & & \overline{M_{Hod,0_{\mathbb{A}^1}}(C/B)} \arrow[ddd, "\overline{h_{Hod,0_{\mathbb{A}^1}}}"] \\ & M_{Dol}(C/B) \arrow[ul] \arrow[r] \arrow[d, "h_{Dol}"'] & M_{Hod,0_{\mathbb{A}^1}}(C/B) \arrow[ur] \arrow[d, "h_{Hod,0_{\mathbb{A}^1}}"] & \\ & A(C/B) \arrow[r, "Fr_A"] \arrow[dl] & A(C^{(B)}/B) \arrow[dr] & \\ \overline{A(C/B)} \arrow[rrr, "Fr_{\overline{A}}"] & & & \overline{A(C^{(B)}/B)}, \end{tikzcd} \tag{12}$$

where

1. *All the oblique arrows in (12) are open immersions with dense image;*
2. *All vertical and horizontal arrows in (12) are projective morphisms;*
3. *The top two horizontal arrows satisfy the properties in Theorem 2.19.(4);*
4. *If we further require that J is algebraically closed, then the top two arrows are universal homeomorphisms;*
5. *$\overline{A}$ is the weighted projective space $\mathbb{P}(1,1,2,\ldots,r)$ associated with the $\mathbb{G}_{m,B}$-variety $\mathbb{A}^1 \times \prod_{i=1}^r \mathbb{A}'_i$, where $\mathbb{G}_{m,B}$ acts as standard dilations of weight 1 on $\mathbb{A}^1$ and of weight i on the remaining factors and, as the notation indicates, the morphism $Fr_{\overline{A}}$ is the relative Frobenius morphism for the B-scheme $\overline{A}$.*

The proofs of Theorems 2.19 and 2.20 are postponed to §4.3.

Remark 2.21. We have borrowed the construction of $\overline{\tau}$ in (9) from [15, Thm. 3.2], where it is proved, over the complex numbers, that Simpson's compactification of the Dolbeault moduli space is projective.

3. Proof of the compactification Theorem I 2.6

The purpose of this section is to prove the Compactification Theorem I 2.6. In §3.1, we prove some well known lemmata that are used in later sections. In §3.2, we construct an object $Z(r)$ that is used in the proofs of all the items of Theorem 2.6. In §3.3, we prove item (1) of Theorem 2.6 which states that Y_+ and Y_- are closed inside X and that the uniform geometric quotient $U/\mathbb{G}_{m,B}$ exists. In §3.4, we prove the item (2) of Theorem 2.6 which states that $U/\mathbb{G}_{m,B}$ exists as a uniform geometric quotient. In §3.5, we prove the item (3) of Theorem 2.6 which states that $U/\mathbb{G}_{m,B} \to S$ is universally closed. In §3.6, we prove the item (4) of Theorem 2.6 which states that $U/\mathbb{G}_{m,B} \to S$ is separated.

3.1. Some preparatory Lemmata

In this subsection, we prove Lemmata 3.1, 3.2, and 3.3. While they are all well-known and quite general, we could not find formal references for the exact statements that we need in the subsequent sections.

We use the following version of valuative criterion in the proof of universal closedness, which follows from [19, Ex.II.4.11.(b)] and the proof of [41, 03K8]:

Lemma 3.1 ***(Valuative Criterion for Universal Closedness)****. Let $f : X \to S$ be a morphism of finite type between noetherian schemes. Then f is universally closed iff given any DVR R_0 inside its fraction field K and a commutative diagram*

$$\begin{tikzcd} Spec(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Spec(R_0) \arrow[r] & Y, \end{tikzcd} \tag{13}$$

we can find a field extension $L \supset K$, a valuation ring R inside L dominating R_0, and a morphism $Spec(R) \to X$, making the following diagram commutative:

$$\begin{tikzcd} Spec(L) \arrow[r] \arrow[d] & Spec(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Spec(R) \arrow[r] \arrow[urr] & Spec(R_0) \arrow[r] & Y. \end{tikzcd} \tag{14}$$

We need two Lemmata about lifting group actions along normalizations and blowing ups in the proof of Lemma 3.5.

Lemma 3.2 ***(Group actions on blowups)****. Let X be a locally noetherian scheme over a scheme T. Let G be a group scheme over T that is locally noetherian. Let $\mu : G \times_T X \to X$ be a G-action on X. Let Y be a G-invariant closed subscheme of X with dense complement. Let $\widetilde{X}$ be the blowing up of X along Y. Suppose that $G \times_T G \times_T \widetilde{X}$, $G \times_T \widetilde{X}$, and $\widetilde{X}$ are reduced, and that X is separated.*

Then the blowing up $\widetilde{X}$ of X along Y admits a G-action making the blow down morphism $\pi : \widetilde{X} \to X$ to be G-equivariant.

Proof. We have the canonical isomorphisms

$$G \times_T Y \cong (G \times_T X) \times_{p_X, X} Y \cong (G \times_T X) \times_{\mu, X} Y, \tag{15}$$

where p_X is the natural projection $G \times X \to X$, the morphisms from Y to X are always the inclusion $Y \hookrightarrow X$, and we have included the morphisms in the subscript to emphasize which fiber product we are taking. Indeed, the first isomorphism is automatic, and the second isomorphism follows from the G-invariance of Y.

Since p_X is flat, by [30, Prop 8.1.12.(c)], the blow up of $G \times_T X$ with center $G \times_T Y$ is canonically isomorphic to $G \times_X \widetilde{X}$. By (15), we see that $G \times_X \widetilde{X}$ is also the blow up of $G \times_T X$ with center the fiber $\mu^{-1}(Y)$. By the universal property of the blow up [30, Prop 8.1.15], there exists a unique morphism $\widetilde{\mu} : G \times_T \widetilde{X} \to G \times_T X$, making the following diagram commutative:

$$\begin{tikzcd} G \times_T \widetilde{X} \arrow[r, "\widetilde{\mu}"] \arrow[d] & \widetilde{X} \arrow[d] \\ G \times_T X \arrow[r, "\mu"'] & X. \end{tikzcd} \tag{16}$$

Let $\mu^G : G \times_T G \to G$ be the group multiplication morphism and $e : T \to G$ be the identity morphism. To verify that $\widetilde{\mu}$ defines a group action on $\widetilde{X}$, and that $\pi : \widetilde{X} \to X$ is G-equivariant, we need to show the following three identities of morphisms:

$$\widetilde{\mu} \circ (1_G \times \widetilde{\mu}) = \widetilde{\mu} \circ (\mu^G \times 1_{\widetilde{X}}) : \; G \times_T G \times_T \widetilde{X} \to \widetilde{X}, \tag{17}$$

$$\widetilde{\mu} \circ (e \times 1_{\widetilde{X}}) = 1_{\widetilde{X}} : \; \widetilde{X} \to \widetilde{X}, \tag{18}$$

$$\mu \circ (1_G \times \pi) = \pi \circ \widetilde{\mu} : \quad G \times_T \widetilde{X} \to X. \tag{19}$$

All three pairs of morphisms agree on the open and dense subscheme corresponding to $X \setminus Y$. By the assumption on the reducedness and separatedness on the domains and target, [19, Ex II.4.2] shows that the three identities hold over all of their domains. (Note that the separatedness of X implies the separatedness of $\tilde{X}$ by [41, 01O2]). $\square$

Lemma 3.3 *(**Group actions on normalization**). Let X be a scheme over a scheme T. Suppose that X is an integral scheme. Let G be a group scheme over T that acts on X via the action morphism $\mu : G \times_T X \to X$. Let $\pi : X' \to X$ be the normalization of X. If $G \times_T X'$ is a normal and integral scheme, then X' admits a G-action, making the normalization morphism $\pi : X' \to X$ to be G-equivariant.*

Proof. Consider the surjective morphism $G \times_T X' \xrightarrow{1_G \times \pi} G \times_T X \xrightarrow{\mu} X$. Since $G \times_T X$ is normal and integral, the universal property of normalization [17, Prop 12.44] induces a unique morphism $\mu' : G \times_T X' \to X'$ so that we have the equality of morphisms:

$$\mu \circ (1_G \times \pi) = \pi \circ \mu' : \quad G \times_T X' \to X. \tag{20}$$

If we can show that μ' is a G-action, then (20) shows that the normalization $\pi : X' \to X$ is G-equivariant. We now proceed to show that μ' is indeed a G-action: Let $\mu^G : G \times_T G \to G$ be the group multiplication morphism and $e : T \to G$ be the identity morphism. We need to show the following two equalities of morphisms:

$$\mu' \circ (1_G \times \mu') = \mu' \circ (\mu^G \times 1_{X'}) : \; G \times_T G \times_T X' \to X', \tag{21}$$

$$\mu' \circ (e \times 1_{X'}) = 1_{X'} : \; X' \to X'. \tag{22}$$

To show (21), by the uniqueness of μ', it suffices to show that the two morphisms are equal after a composition of $\pi : X' \to X$, i.e., we want to show that

$$\pi \circ \mu' \circ (1_G \times \mu') = \pi \circ \mu' \circ (\mu^G \times 1_{X'}) : \; G \times_T G \times_T X' \to X' \to X. \tag{23}$$

By (20), both the morphisms in (23) factors as

$$G \times_T G \times_T \times X' \xrightarrow{1_G \times 1_G \times \pi} G \times_T G \times_T \times X \to X, \tag{24}$$

where the last morphism is

$$\mu \circ (1_G \times \mu) = \mu \circ (\mu^G \times 1_X) : \ G \times_T G \times_T X \to X. \tag{25}$$

We thus have (21). The equality (22) follows similarly: by (20), we have

$$\pi \circ \mu' \circ (e \times 1_{X'}) = \mu \circ \pi \circ (1_G \times \pi) \circ (e \times 1_{X'}) = \mu' \circ (e \times 1_X) = 1_X, \tag{26}$$

so we have that (22) holds after composing π, hence (22) holds by the uniqueness of f' in the universal property of normalization. □

3.2. The schemes $Z(r)$

Our goal in §3 is to prove Theorem 2.6.

A feature of our proof of Theorem 2.6 is that the proof of Y_+ and Y_- are closed is similar to the proof of the universal closedness of $U/\mathbb{G}_{m,B} \to S$, in the sense that both proofs start with a point $r \in X(R)$ for some discrete valuation ring R, which then produces a rational map $\mathbb{P}^1_R \dashrightarrow X_R$, and both proofs rely heavily on the structure of the resolution of indeterminancy $Z(r)$ of the rational map $\mathbb{P}^1_R \dashrightarrow X_R$. Therefore it seems best to first introduce and study $Z(r)$ in this Section 3.2 and then to diverge to separate proofs of the items (1)-(2) of Theorem 2.6 in Sections 3.3-3.6.

Our next goal is to define what $Z(r)$ is. We do so in Lemma 3.4.

Let R be a discrete valuation ring with fraction field L and residue field κ.

Let $r \in X(R)$. Let $\eta \in X(L)$ be the restriction of r to the open subscheme $\mathrm{Spec}(L) \subset \mathrm{Spec}(R)$. Taking the orbit of η, we have the morphism $\mu_\eta : \mathbb{G}_{m,L} \to X$. Since $\mathbb{G}_{m,B}$ acts trivially on S, we have that the image of the composition $\mathbb{G}_{m,L} \xrightarrow{\mu_\eta} X \to S$ is a point, and that the morphism $\mathbb{G}_{m,R} \xrightarrow{\mu_\eta} X \to S$ factors through a morphism $\mathrm{Spec}(R) \to S$. Therefore we can extend the morphism $\mathbb{G}_{m,L} \to S$ to a morphism $\mathbb{P}^1_L \to S$. Since X is proper over S, we can extend μ_η to an S-morphism $\overline{\mu_\eta} : \mathbb{P}^1_L \to X$. We then have a graph morphism $\Gamma_{\overline{\mu_\eta}} : \mathbb{P}^1_L \to \mathbb{P}^1_L \times_S X$, which is a closed immersion [19, p.106], and is $\mathbb{G}_{m,L}$-equivariant. Let $j : \mathbb{P}^1_L \times_S X \to \mathbb{P}^1_R \times_S X$ be the natural open immersion induced by the open immersion $\mathrm{Spec}(L) \hookrightarrow \mathrm{Spec}(R)$. Let W be the scheme theoretic image of $j \circ \Gamma_{\overline{\mu_\eta}} : \mathbb{P}^1_L \to \mathbb{P}^1_R \times_S X$. Since X is projective over S, we have that $X_R := X \times_S \mathrm{Spec}(R)$ is projective over $\mathrm{Spec}(R)$. The morphism $p_X \circ j \circ \Gamma_{\overline{\mu_\eta}} : \mathbb{P}^1_L \to X$ induces a rational map $b : \mathbb{P}^1_R \dashrightarrow X_R$.

Lemma 3.4 *(**Introduce** $Z(r)$). There exists a proper birational morphism $\pi_5 : Z(r) \to W$ with $Z(r)$ regular, making the following diagram commutative:*

$$\begin{tikzcd} \mathbb{P}^1_R \arrow[d, "\pi_2"'] \arrow[dr, dashed, "b"] & W \arrow[l, "\pi_1"'] \arrow[d, "\pi_3"] & Z(r) \arrow[l, "\pi_5"'] \\ R & X_R. \arrow[l, "\pi_4"] \end{tikzcd} \tag{27}$$

We have that π_5 is an isomorphism above every regular point of W. Moreover,

(1) $Z(r)$ is the last element W_n in the following sequence:

$$\pi_5 : Z(r) = W_n \to W_{n-1} \to \dots \to W_1 \to W_0 = W, \tag{28}$$

where $W_1 \to W$ is the normalization, and for every $i \geq 1$, $W_{i+1} \to W_i$ is obtained by first blowing up, $\widetilde{W_i} \to W_i$, the singular locus (which by definition is reduced) of W_i, and then by normalizing, $W_{i+1} \to \widetilde{W_i}$, the resulting $\widetilde{W_i}$;

(2) $Z(r)$ is also the last element Z_n in the following sequence

$$\pi_6 : Z(r) = Z_n \xrightarrow{p_n} Z_{n-1} \xrightarrow{p_{n-1}} \dots \xrightarrow{p_1} Z_0 = \mathbb{P}^1_R, \tag{29}$$

where for each $i \geq 1$, Z_i is obtained by blowing up a closed point of Z_{i-1}.

Proof. The commutative diagram (27) is exactly the elimination of points of indeterminacy for the rational map b as constructed in [30, Thm. 9.2.7], where it is shown that every rational map from a regular fibered surface over a one dimensional Dedekind scheme D (such as $\mathbb{P}^1_R$ over $\mathrm{Spec}(R)$) to a projective D-scheme (such as X_R over $\mathrm{Spec}(R)$) admits an elimination of indeterminacy $\pi_6 : Z(r) \to Z_0$, which is a finite sequence of blowing-ups of closed points of the target as in (29), and which factors through the desingularization of the closure of the graph of the rational map as in (28). We thus obtain our lemma as a direct application of [30, Thm. 9.2.7]. □

Our goal in the remainder of this §3.2 is to prove Lemma 3.7, which describes the reduction of the closed fiber of $Z(r)$ over $\mathrm{Spec}(R)$. We now fix $r \in X(R)$, and suppress the argument r in $Z(r) = Z$. We start with the following

Lemma 3.5 (**$\mathbb{G}_{m,R}$ action on the partial resolutions** Z_i)**.** *Each Z_i, $0 \leq i \leq n$, in the sequence (29) admits a $\mathbb{G}_{m,R}$-action so that each $p_i : Z_i \to Z_{i-1}$ is $\mathbb{G}_{m,R}$-equivariant. Furthermore, each $p_{i+1} : Z_{i+1} \to Z_i$ is a blow up of a $\mathbb{G}_{m,R}$-fixed closed point of Z_i.*

Proof. We first show the

CLAIM: $Z_n = Z$ admits a $\mathbb{G}_{m,R}$-action so that $\pi_6 : Z \to \mathbb{P}^1_R$ is $\mathbb{G}_{m,R}$-equivariant.

Then we finish the proof using an increasing induction on $i \geq 0$.

To show the **CLAIM** above, by Lemmata 3.2, 3.3, and 3.4.(1), we see that it suffices to show that for each $i \geq 1$ the singular locus of W_i is made of $\mathbb{G}_{m,R}$-fixed points. Indeed, since each W_i, $i \geq 1$, is normal, the singular locus has to be closed points [30, Prop. 4.2.24]. If these singular closed points are not fixed by the $\mathbb{G}_{m,R}$-action, then the orbits of the points under $\mathbb{G}_{m,R}$ would form a one dimensional subscheme of the singular locus of W_i, contradicting the normality of W_i. The **CLAIM** is thus proved.

We now prove the lemma by induction on $i \geq 0$.

Base case $i = 0$:

The $\mathbb{G}_{m,R}$-action on $\mathbb{P}^1_R$ is the natural one induced by the multiplication on the open subscheme $\mathbb{G}_{m,R} \subset \mathbb{P}^1_R$. The statement about p_0 is vacuous. Let z_0 be the closed point of Z_0 that is the center of the blow up $p_1 : Z_1 \to Z_0$. We would like to show that z_0 is a $\mathbb{G}_{m,R}$-fixed point:

If z_0 is not fixed by the $\mathbb{G}_{m,R}$-action, then the fiber $\pi_6^{-1}(z_0)$ inside Z has an irreducible component E that is not stable under the $\mathbb{G}_{m,R}$-action. Let ξ_E be the generic point of E. Consider the orbit morphism $\mu_{\xi_E} : \mathbb{G}_{m,k(\xi_E)} \to Z$. The scheme theoretic image $O(E)$ of μ_{ξ_E} properly contains E as a closed subscheme since E is not $\mathbb{G}_m$-stable. Thus $\dim(O(E)) \geq 2$. Since Z is two dimensional and integral, the set theoretic image of μ_{ξ_E} is dense inside Z. Therefore, the set theoretic image $\pi_6(Z)$ is contained in the closure of the $\mathbb{G}_{m,R}$-orbit of z_0, which is one dimensional, contradicting that π_6 is surjective. Thus z_0 has to be a $\mathbb{G}_{m,R}$-fixed point, and the base case is established.

Now suppose that we have established the case $i-1$ and would like to show the case i.

Since $p_i : Z_i \to Z_{i-1}$ is a blow up of a $\mathbb{G}_{m,R}$-fixed closed point of Z_{i-1}. By Lemma 3.2, we see that Z_i admits a $\mathbb{G}_{m,R}$-action so that p_i is $\mathbb{G}_{m,R}$-equivariant. Let z_i be the closed point of Z_i that is the center of the blow up $p_{i+1} : Z_{i+1} \to Z_i$. We would like to show that z_i is a $\mathbb{G}_{m,R}$-fixed point. We first show that the morphism $p_n \circ ... \circ p_{i+1} : Z \to Z_i$ is $\mathbb{G}_{m,R}$-equivariant, i.e., the analogue of the equation (19) holds in our case:

$$\alpha_{Z_i} \circ (1_G \times (p_n \circ ... \circ p_{i+1})) = p_n \circ ... \circ p_{i+1} \circ \alpha_Z : \quad \mathbb{G}_{m,R} \times_R Z \to Z_i, \tag{30}$$

where α_Z and α_{Z_i} denote the $\mathbb{G}_{m,R}$-action morphisms on Z and Z_i. Since Z_i is obtained from $Z_0 = \mathbb{P}^1_R$ by iterated blowups on closed points, by the inductive hypothesis we have that the projection $p_i \circ ... \circ p_1 : Z_i \to Z_0$ is a $\mathbb{G}_{m,R}$-equivariant isomorphism over an open and dense subscheme U_0 of Z_0. From the **CLAIM**, we see that the equation (30) holds when restricted to the open and dense subscheme $\mathbb{G}_{m,R} \times_R \pi_6^{-1}(U_0)$ of $\mathbb{G}_{m,R} \times_R Z$. Therefore, by [19, Ex II.4.2], we see that the equality (30) holds, so $p_n \circ ... \circ p_{i+1}$ is $\mathbb{G}_{m,R}$-equivariant. We can now use the argument in the base case to conclude. Namely, if z_i is not fixed by the $\mathbb{G}_{m,R}$-action, then $(p_n \circ ... \circ p_{i+1})^{-1}$ would trace out a dense two dimensional subscheme of Z under the $\mathbb{G}_{m,R}$-action. We then have that $p_n \circ ... \circ p_{i+1} : Z \to Z_i$ maps Z to the $\mathbb{G}_{m,R}$-orbit of z_i in Z_i, which is one dimensional, contradicting that $\pi_6 : Z \to Z_0$ is surjective. $\square$

Lemma 3.5 above essentially contains all the information of Z in Lemma 3.7. However, in order to keep track of the $\mathbb{G}_{m,R}$-actions under each blow up $p_i : Z_i \to Z_{i-1}$ in the proof of Lemma 3.7, we need to study a family of affine charts involved in the sequence of blow ups (29). The subtlety is that different $\mathbb{G}_{m,R}$-fixed points of Z_i have non-isomorphic, although similar, $\mathbb{G}_{m,R}$-invariant affine neighborhood, and that we need to know what happens to the $\mathbb{G}_{m,R}$-action when we blow up any of the $\mathbb{G}_{m,R}$-fixed points of Z_i.

We start with the affine charts for $Z_0 = \mathbb{P}^1_R$: we have that A^1_R and $\mathbb{P}^1_R \setminus 0_R \cong A^1_R$ cover Z_0, so the two affine charts are isomorphic to $\mathrm{Spec}(R[x])$ and $\mathrm{Spec}(R[x^{-1}])$, where x is an independent variable. The variable x has positive weight $w(x)$ under the $\mathbb{G}_{m,R}$-action. Let λ be the uniformizing parameter of R, by the triviality of the $\mathbb{G}_{m,R}$-action on $\mathrm{Spec}(R)$, we have that the weight of λ under $\mathbb{G}_{m,R}$ is $w(\lambda) = 0$.

The blow up $p_1 : Z_1 \to Z_0$ has center 0_κ or ∞_κ. Without loss of generality, assume p_1 is the blow up of 0_κ. (If p_1 is the blow up of ∞_κ then we can exchange x and x^{-1}). By the description of a blow up algebra of a regular algebra in [30, p.325 bottom], we have that Z_1 can be covered by three $\mathbb{G}_{m,R}$-invariant affine charts isomorphic to $\mathrm{Spec}(R[x, \lambda/x])$, $\mathrm{Spec}(R[y])$ and $\mathrm{Spec}(R[x^{-1}])$, where y is a new independent variable. We have that the weight of x under the $\mathbb{G}_{m,R}$-action is still $w(x)$, while the weight of λ/x under the $\mathbb{G}_{m,R}$-action is $w(\lambda/x) = w(\lambda) - w(x) = -w(x)$. Since the exceptional divisor of p_1 is defined by y in $\mathrm{Spec}(R[y])$ and λ/x in $\mathrm{Spec}(R[x, \lambda/x])$, we have that the weight of y under the $\mathbb{G}_{m,R}$-action is $w(y) = -w(\lambda/x) = w(x)$.

Since $w(\lambda/x), w(x) \neq 0$, we have that the $\mathbb{G}_{m,R}$-fixed point of $\mathrm{Spec}(R[x, \lambda/x])$ is defined by the maximal ideal $\langle x, \lambda/x\rangle$ generated by x and $\frac{\lambda}{x}$. The blow up of $\mathrm{Spec}(R[x, \lambda/x])$ with center $\langle x, \lambda/x\rangle$ is covered by the $\mathbb{G}_{m,R}$-invariant affine charts $\mathrm{Spec}(R[x, \frac{\lambda}{x^2}])$ and $\mathrm{Spec}(R[\frac{x^2}{\lambda}, \frac{\lambda}{x}])$. As for the weight under the $\mathbb{G}_{m,R}$-action, we have that $w(\lambda/x^2) = -2w(x)$.

Writing out the charts for iterated blow ups of $\mathrm{Spec}(R[x, \lambda/x])$ at $\mathbb{G}_m$-fixed points as above, it is easy to see that we have the following

Lemma 3.6 ***(Compatibility of the weights at intersection points)***. *For each Z_i, $0 \le i \le n$, in the sequence (29), we have that Z_i can be covered by affine charts that are isomorphic to Spec$(R[x])$ or*

$$\mathit{Spec}(R[z_1, z_2]/(z_1^a z_2^b - \lambda)), \tag{31}$$

for some $a, b \in \mathbb{Z}_{\ge 0}$, with $a + b > 0$.

The weight of x under the $\mathbb{G}_{m,R}$-action is nonzero. Let $w(z_1)$ and $w(z_2)$ be the weights of z_1 and z_2 under the $\mathbb{G}_{m,R}$-action, then we have that $w(z_1)$ and $w(z_2)$ are both nonzero, and that $aw(z_1) + bw(z_2) = 0$. In particular, we have that $w(z_1)w(z_2) < 0$.

Proof. From the paragraphs above Lemma 3.6, we see that Z_i is covered by affine charts that are isomorphic to $\mathrm{Spec}(R[x])$ and iterated blow ups of $\mathrm{Spec}(R[x, \lambda/x])$ at $\mathbb{G}_m$-fixed points. We would like to show that such blow ups can be covered by the charts of the form as in (31), and that the weights satisfies what in the statement of this Lemma 3.6.

We show this by induction on the number of blow ups of $\mathrm{Spec}(R[x, \lambda/x])$. The base case, where there are no blow ups, is satisfied because we can take $z_1 = x$, $z_2 = \lambda/x$,

$a = b = 1$, and we have $w(x) + w(\lambda/x) = 0$. For the inductive step, we blow up a $\mathbb{G}_m$-fixed point of the chart in (31). Since $w(z_1), w(z_2) \neq 0$, the $\mathbb{G}_m$-fixed point has to be the maximal ideal $\langle z_1, z_2 \rangle$. The resulting blow up can be covered by the spectra of $R[z_1, \frac{z_2}{z_1}]$ and $R[\frac{z_1}{z_2}, z_2]$, with $z_1^a z_2^b = \lambda$, i.e., the spectra of

$$R[z_1', z_2']/((z_1')^{a+b}(z_2')^b - \lambda) \text{ and } R[z_1'', z_2'']/((z_1'')^a(z_2'')^{a+b} - \lambda). \tag{32}$$

Furthermore, we have that

$$w(z_1') = w(z_1),\ \ w(z_2') = w(z_2) - w(z_1),\ \ w(z_1'') = w(z_1) - w(z_2),\ \ w(z_2'') = w(z_2). \tag{33}$$

Now it is easy to check that the weights have the desired properties in the statement of the lemma. $\square$

We can now prove the main Lemma 3.7 as a corollary of Lemma 3.6:

Lemma 3.7 *(**Shape of Closed Fiber of** Z_i **over Spec**(R)). For each Z_i, $0 \leq i \leq n$, in the sequence (29), let*

$$E_i := (Z_i \times_R \kappa)_{red} \tag{34}$$

be the reduction of the fiber of $Z_i \to Spec(R)$ over the closed point $Spec(\kappa) \hookrightarrow Spec(R)$. We have that for each $0 \leq i \leq n$,

(1) E_i is connected;
(2) the irreducible components of E_i are all isomorphic to $\mathbb{P}^1_\kappa$;
(3) the singular points of E_i are where the irreducible components of E_i meet, and the singular points are all ordinary double points;
(4) all of the irreducible components of E_i admit nontrivial $\mathbb{G}_{m,\kappa}$-actions so that every singular point of E_i, which lies in only two irreducible components of E_i, is the 0-limit of one component and the ∞-limit of the other component;
(5) the relation $\leq$ in Definition 2.2 gives a linear order on the set of $\mathbb{G}_{m,\kappa}$-fixed points of E_i. Furthermore, there are $i+1$ $\mathbb{G}_{m,\kappa}$-fixed points, $i-1$ of which are singular, and the two regular $\mathbb{G}_{m,\kappa}$-fixed points are the unique maximal and minimal elements with respect to $\leq$.

Remark 3.8 *(**Lemma 3.7 in terms of graphs**).* For each $0 \leq i \leq n$, define a directed graph $\Gamma(E_i)$ as follows: with each irreducible component of E_i we associate an edge, and with each $\mathbb{G}_{m,\kappa}$-fixed point of E_i we associate a vertex; according to Lemma 3.7.(2),(4), each irreducible component $\mathcal{E}$ of E_i contains two $\mathbb{G}_{m,\kappa}$-fixed points e_1 and e_2, which are the 0 and ∞ limits of the $\mathbb{G}_{m,\kappa}$-action on $\mathcal{E}$ respectively, thus we can let the vertices corresponding to e_1 and e_2 be the source and the end of the edge $ed(\mathcal{E})$ corresponding to $\mathcal{E}$ respectively, and let the direction on $ed(\mathcal{E})$ be pointing from the source to the end. Then Lemma 3.7 implies that $\Gamma(E_i)$ has $i+1$ vertices, and is of the form

$$\circ \to \circ \to \ldots \to \circ.$$

Proof of Lemma 3.7. Since $p_i : Z_i \to Z_{i-1}$ is the blow up of a closed point of Z_{i-1}, we have that $E_i = (p_i^{-1}(E_{i-1}))_{red}$ is connected if and only if E_{i-1} is connected. Since $E_0 = \mathbb{P}^1_\kappa$ is connected, we have that each E_i is connected, thus we have item (1).

From Lemma 3.6, we see that the fiber $Z_i \times_R \kappa$ can be covered by the affine charts that are isomorphic to the spectra of $\kappa[x]$ or $\kappa[z_1, z_2]/(z_1^a z_2^b)$ for some $a, b \in \mathbb{Z}_{\geq 0}$ with $a + b > 0$. Therefore E_i can be covered by the affine charts that are isomorphic to the spectra of $\kappa[x]$ or $\kappa[z_1, z_2]/(z_1 z_2)$. Items (2) and (3) are immediate from these charts.

From the description of weights in Lemma 3.6, we see that x has nontrivial weights, and that $w(z_1)w(z_2) < 0$. Therefore we have item (4).

We prove item (5) by an induction on $i \geq 0$. The base case $i = 0$ is automatic because we have that two regular $\mathbb{G}_{m,\kappa}$-fixed points 0_κ and ∞_κ of $E_0 = \mathbb{P}^1_\kappa$, and that $0_\kappa \leq \infty_\kappa$. Now suppose we have proved the case $i-1$. Let us order the i $\mathbb{G}_{m,\kappa}$-fixed points of E_{i-1} as $z_1 \leq \ldots \leq z_i$. Suppose $p_i : Z_i \to Z_{i-1}$ is the blow up with center z_j for some $1 \leq j \leq i$. We then have that p_i is a $\mathbb{G}_{m,R}$-equivariant isomorphism when restricted to the affine charts for $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_i$ selected in Lemma 3.6. For each $k \neq j$, let $y_k := p^{-1}(z_k)$. Let y_j^0 and y_j^∞ be the 0 and ∞ point of the exceptional divisor, which is isomorphic to $\mathbb{P}^1_\kappa$ by item (2), of the blow up p_i. It then follows from item (4) that we can linearly order the $\mathbb{G}_m$-fixed points of E_i as $y_1 \leq \ldots \leq y_{j-1} \leq y_j^0 \leq y_j^\infty \leq y_{j+1} \leq \ldots \leq y_i$. We have thus showed the first sentence of item (5) in case i. Since each of y_1 and y_i lies in only one component of E_i, from Lemma 3.6 we see that y_1 and y_i have affine neighborhoods isomorphic to $\mathrm{Spec}(R[x])$, thus y_1 and y_i are regular. Any $\mathbb{G}_{m,\kappa}$-fixed point of E_i that is not y_1 or y_i lies in two components of E_i, thus cannot be regular. We have thus showed the case i of item (5). $\square$

3.3. Closedness of Y_+ and Y_-

In this Section 3.3, we prove Theorem 2.6.(1) which states that both Y_- and Y_+ are closed inside X.

The sets Y_+ and Y_- are constructible (see the first paragraph of Section 3.2), thus it suffices to show that Y_+ and Y_- are closed under specializations. Let x, x' be two Zariski points of X so that $x' \in \overline{\{x\}}$. Assume that $x \in Y_+$. We would like to show that $x' \in Y_+$.

By [19, Lemma II.4.4, Ex. II.4.11], there exists a discrete valuation ring R with fraction field L and residue field κ, and an R-point $r \in X(R)$, so that r maps the generic point of $\mathrm{Spec}(R)$ to x and the closed point of $\mathrm{Spec}(R)$ to x'.

Take $Z = Z(r)$ as defined in Lemma 3.4. By Lemma 3.5 we have that the morphism $\pi_6 : Z \to \mathbb{P}^1_R$ is the composition of iterated $\mathbb{G}_{m,R}$-equivariant blow ups at $\mathbb{G}_{m,R}$-fixed closed point. Therefore π_6 restricted over the closed subscheme $1_R \subset \mathbb{P}^1_R$ is an isomorphism. Thus $\pi_6^{-1}(1_R)$ is an R-point of Z. We employ the notation as in (27). Define the composition of $\mathbb{G}_{m,R}$-equivariant morphisms $\pi_7 : Z \xrightarrow{\pi_5} W \xrightarrow{\pi_3} X_R \xrightarrow{p_X} X$. We then have that $\pi_7|_{\pi_6^{-1}(1_R)} = r : \mathrm{Spec}(R) \to X$. We then have an equality of L-points of X:

$$\pi_7(\pi_6^{-1}(\infty_L)) = \lim_{t\to\infty} t \cdot x. \tag{35}$$

We employ the notation in Lemma 3.7. By keeping track of what happens to the affine charts containing ∞_L in $\mathbb{P}^1_L$ under each blow up $p_i : Z_i \to Z_{i-1}$ as in Lemma 3.6, we see that $\pi_6^{-1}(\infty_L)$ in Z specializes to the maximal element z_{n+1} with respect to the linear order $\leq$. By (35), we have that $\pi_7(z_{n+1}) \in \overline{\{\lim_{t\to\infty} t \cdot x\}}$. Since $x \in Y_+$, and V_+ is closed, we have that $\pi_7(z_{n+1}) \in V_+$. Since π_7 is $\mathbb{G}_{m,B}$-equivariant, for any $0 \leq j \leq n+1$, we have that $\pi_7(z_j) \leq \pi_7(z_{n+1})$. By the defining property of V_+, we see that $\pi_7(z_j) =\in V_+$. In particular, we can let z_j be the ∞-limit of $\pi_6^{-1}(1_\kappa)$, we then have that

$$\lim_{t\to\infty} t \cdot x' = \lim_{t\to\infty} t \cdot r|_{\mathrm{Spec}(\kappa)} = \lim_{t\to\infty} t \cdot \pi_7|_{\pi_6^{-1}(1_\kappa)} = \pi_7(z_j) \in V_+. \tag{36}$$

Therefore we have that $x' \in Y_+$, and we have proved that Y_+ is closed inside X.

The proof that Y_- is closed inside X is very similar to the proof above, except that we exchange 0 and ∞ in the argument. We have thus proved Theorem 2.6.(1).

3.4. Existence of uniform geometric quotient $U/\mathbb{G}_{m,B}$

In this section, we prove Theorem 2.6.(2) which states that a uniform geometric quotient $\phi : U \to U/\mathbb{G}_{m,B}$ exists.

Proof of Theorem 2.6.(2). By Assumption 2.1, we can cover U with $\mathbb{G}_{m,B}$-invariant open affine subschemes $U = \bigcup_i Q_i$. By [37, Thm. 3, Rmk. 8, 10], for each i, there exists a uniform categorical quotient $\phi_i : Q_i \to Q_i/\mathbb{G}_{m,B}$ which is surjective, and that $Q_i/\mathbb{G}_{m,B}$ is of finite type over B.

We first show that these ϕ_i's glue to form a uniform categorical quotient $\phi : U \to U/\mathbb{G}_{m,B}$, which is immediately reduced to showing that for every i, j, we have that $Q_i \cap Q_j = \phi^{-1}\phi_i(Q_i \cap Q_j)$, see the proof of [1, Prop. 7.9]. Since any two closed point q^i_1, q^i_2 in Q_i are mapped to the same point in $Q_i/\mathbb{G}_{m,B}$ if and only if the closures (inside Q_i) of the orbits of q^i_1 and q^i_2 intersect nontrivially, see [37, Thm. 3.(ii)], we are further reduced to showing that every closed point q^i in Q_i has closed orbit in Q_i:

Since the closure of the orbit of q^i in X has boundaries contained in V, we have that the orbit of q^i is closed inside $X - V$. By Theorem 2.6.(1), we have that U is open inside X, thus the orbit of q^i is closed inside U, thus closed inside Q_i. Therefore, we have shown that there exists a uniform categorical quotient $\phi : U \to U/\mathbb{G}_{m,B}$.

We now show that ϕ is indeed a uniform geometric quotient:

[37, Thm. 3.(iii)] shows that the image of any $\mathbb{G}_{m,B}$-stable closed subscheme of U is closed in $U/\mathbb{G}_{m,B}$, which, combined with the proof of [16, p.8, Rmk. (6)], shows that ϕ

is submersive. Since $\mathbb{G}_{m,B}$ is open over B, by [26, Rmk. 2.8.3], we have that ϕ is actually universally submersive.

Checking the definition of uniform geometric quotients as in [16, Def. 0.6], we have that to conclude the proof, it suffices to show that for any algebraically closed field K and a point K-point $a \in U/\mathbb{G}_{m,B}(K)$, the fiber $\phi^{-1}(a)$ contains only one $\mathbb{G}_{m,B}$-orbit:

If the dimension of $\phi^{-1}(a)$ is larger than 1, then by generic flatness, there exists a closed point $u \in \overline{\{a\}}$ such that $\phi^{-1}(u)$ has dimension larger than 1, which contradicts the fact that a closed point in U has a closed orbit in U, which is established above.

Therefore, it remains to show that $\phi^{-1}(a)$ is irreducible:

Again, by [37, Thm. 3.(ii)], any two geometric points of U are mapped to the same geometric point of $U/\mathbb{G}_{m,B}$ by ϕ if and only if the closures (inside U) of the orbits of the two geometric points intersect nontrivially. We have seen above that any closed point u in U has a closed orbit in U, thus $\phi^{-1}(u)$ is irreducible. By [41, 0553], we must have that $\phi^{-1}(a)$ is irreducible for any geometric point of $U/\mathbb{G}_{m,B}$. We have that finished the proof. $\square$

Remark 3.9 *($U/\mathbb{G}_{m,B}$ **is a tame and good moduli space**).* Let us verify that, by using the terminology as in [1], the uniform geometric quotient $U/\mathbb{G}_{m,B}$ is a tame and good moduli space for the quotient stack $[U/\mathbb{G}_{m,B}]$:

Since a geometric point in $(X - V)(K)$, where K is an algebraically closed field, has closed orbits in $(X-V)_K$, we have that $X-V$ is the prestable locus for the $\mathbb{G}_{m,B}$-action on X, see [1, Def. 10.1]. By [1, Prop. 11.4], there exists a tame and good moduli space $[(X-V)/\mathbb{G}_{m,B}] \to (X-V)/\mathbb{G}_{m,B}$. By Theorem 2.6.(1), we have that U is open inside $X-V$. Thus $[U/\mathbb{G}_{m,B}]$ is an open substack of $[(X-V)/\mathbb{G}_{m,B}]$, see [21, Rmk. 2.3.1]. Therefore, by [1, Rmk. 7.3, Prop. 7.10], we have that $[U/\mathbb{G}_{m,B}] \to U/\mathbb{G}_{m,B}$ is also a tame and good moduli space.

3.5. Universal closedness of $U/\mathbb{G}_{m,B} \to S$

In this section, we prove Theorem 2.6.(3) which states that the morphism $U/\mathbb{G}_{m,B} \to S$ is universally closed.

Proof of Theorem 2.6.(3). Let us start with the following commutative diagram, where R is a discrete valuation ring, and L is its fraction field

$$\begin{tikzcd} \mathrm{Spec}(L) \arrow[r, "\xi"] \arrow[d] & U/\mathbb{G}_{m,B} \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & S. \end{tikzcd} \tag{37}$$

Let $U^L := U \times_{U/\mathbb{G}_{m,B}} \mathrm{Spec}(L)$. Since $U \to U/\mathbb{G}_{m,B}$ is a geometric quotient, by [16, Def. 0.6.(ii)], we have that the orbit morphism $\mu_\xi : \mathbb{G}_{m,L} \to U$ factors through a surjective morphism $\mu'_\xi : \mathbb{G}_{m,L} \to U^L$. Define $\alpha_1 := \mu'_\xi(1_L)$. The morphisms $\mathbb{G}_{m,L} \xrightarrow{\mu'_\xi} U^L \to \mathrm{Spec}(L)$ give field extensions $L \subset k(\alpha_1) \subset L$. Therefore we have that $\alpha_1 \in U^L(L)$. We now have the following commutative diagram:

$$\begin{tikzcd}
& U^L \arrow[r, shift right] \arrow[d, "\alpha_2"'] & \mathrm{Spec}(L) \arrow[l, bend right, "\alpha_1"'] \arrow[r] \arrow[d, "\xi"] \arrow[dl, "\eta"'] & \mathrm{Spec}(R) \arrow[d] \\
X & U \arrow[l, hook'] \arrow[r] & U/\mathbb{G}_{m,B} \arrow[r] & S.
\end{tikzcd} \tag{38}$$

The composition $\alpha_1 \circ \alpha_2 =: \eta$ defines an L-point $\eta \in U(L)$. By the properness of X/S, we have that η induces an R-point $r \in X(R)$ filling in the commutative diagram (38).

Let $Z = Z(r)$ be as defined in Lemma 3.4. Recall that $\pi_6 : Z \to \mathbb{P}^1_R$ is the composition of iterated $\mathbb{G}_{m,R}$-equivariant blow ups $p_i : Z_i \to Z_{i-1}$ at $\mathbb{G}_{m,R}$-fixed points of Z_{i-1}. From Lemma 3.6, we see that the centers of all blow ups are in the closed fiber $Z_{i-1} \times_R \kappa$. Therefore we have that π_6 is an isomorphism when restricted over the open subscheme $\mathbb{P}^1_L \subset \mathbb{P}^1_R$. Recall that we have the composition of $\mathbb{G}_{m,R}$-equivariant morphism $\pi_7 : Z \xrightarrow{\pi_5} W \xrightarrow{\pi_3} X_R \xrightarrow{p_X} X$. We have that $\pi_7|_{\pi_6^{-1}(1_L)} = \eta : \mathrm{Spec}(L) \to X$, and we have the equalities of L-points of X:

$$\pi_7(\pi_6^{-1}(\infty_L)) = \lim_{t\to\infty} t \cdot \eta, \text{ and } \pi_7(\pi_6^{-1}(0_L)) = \lim_{t\to 0} t \cdot \eta. \tag{39}$$

Since $\eta \in U(L)$, we have that $\lim_{t\to\infty} t \cdot \eta \in V_-(L)$ and $\lim_{t\to 0} t \cdot \eta \in V_+(L)$. We employ the notation in Lemma 3.7. By keeping track of what happens to the affine charts containing 0_L and ∞_L under each blow up as in Lemma 3.6, we see that $\pi_6^{-1}(\infty_L)$ specializes to the maximal element z_{n+1} with respect to the linear order $\leq$ in Definition 2.2, and that $\pi_6^{-1}(0_L)$ specializes to the minimal element z_1 with respect to $\leq$. By (39), we have that $\pi_7(z_{n+1}) \in \overline{V_-} = V_-$, and that $\pi_7(z_1) \in \overline{V_+} = V_+$. Thus there exists a smallest number $1 \leq j \leq n$ so that $\pi_7(z_j) \in V_+$ and $\pi_7(z_{j+1}) \in V_-$. In particular, we have that for any $j' < j$, both $\pi_7(z_{j'})$ and $\pi_7(z_{j'+1})$ are in V_+. Also, for any $j'' > j$, we have that $z_{j''} \geq z_{j+1}$, thus both $\pi_7(z_{j''})$ and $\pi_7(z_{j''+1})$ are in V_-. Therefore j is the unique number so that $\pi_7(z_j) \in V_+$ and $\pi_7(z_{j+1}) \in V_-$.

Let $\mathcal{E}$ be the irreducible component of E_n that contains both z_j and z_{j+1}. By Lemma 3.7, there is an isomorphism $iso : \mathcal{E} \xrightarrow{\sim} \mathbb{P}^1_\kappa$ sending z_j to 0_κ and z_{j+1} to ∞_κ. Let δ_κ be the κ-point of $\mathcal{E}$ so that $iso(\delta_\kappa) = 1_\kappa$. By [30, Lemma 8.3.35.(a)], there exists a Weil divisor Δ of Z that maps surjectively onto $\mathrm{Spec}(R)$ under the projection $Z \to \mathrm{Spec}(R)$, and contains δ_κ. Therefore the fraction field L' of the generic point of Δ is a finite field extension of L. Let $R' \subset L'$ be a DVR dominating R and let $\delta_{R'} \in Z(R')$, so that $\delta_{R'}$ sends the closed point of $\mathrm{Spec}(R')$ to $\delta_\kappa \in E_n(\kappa)$, and the generic point of $\mathrm{Spec}(R')$ to the generic point of Δ. Let $\delta_{L'}$ be the restriction of $\delta_{R'}$ to the generic point $\mathrm{Spec}(L') \subset \mathrm{Spec}(R')$. Since $\pi_7(\delta_\kappa) \in X(\kappa)$ has 0-limit in V_+ and ∞-limit in V_-, we have that $\pi_7(\delta_\kappa) \in U(\kappa)$. Therefore we have that $\delta_{L'} \in \pi_6^{-1}(\mathbb{G}_{m,L})(L')$, so $\delta_{L'}$ and $\pi_6^{-1}(1_L)$ are in the same $\mathbb{G}_{m,L'}$-orbit, thus $\pi_7(\delta_{L'})$ and $\pi_7(\pi_6^{-1}(1_L)) = \eta$ are in the same $\mathbb{G}_{m,L'}$-orbit in X. Since U is $\mathbb{G}_{m,B}$-invariant, and $\eta \in U(L)$, we have that $\pi_7(\delta_{L'}) \in U(L')$. Therefore we have that $\pi_7(\delta_{R'}) \in U(R')$.

We have found $\pi_7(\delta_{L'}) \in U(L')$ which is in the same $\mathbb{G}_{m,L'}$ orbit as η, and which can be extended into $\pi_7(\delta_{R'}) \in U(R')$. In particular, under the quotient morphism $U \to U/\mathbb{G}_{m,B}$, we have that $\pi_7(\delta_{L'})$ is sent to $\xi \in U/\mathbb{G}_{m,B}(L)$ that appears in our starting diagram (37).

Consider the composition $\xi_{R'} : \mathrm{Spec}(R') \xrightarrow{\pi_7 \circ \delta_{R'}} U \to U/\mathbb{G}_m$. We have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Spec}(L') \arrow[r] \arrow[d] & \mathrm{Spec}(L) \arrow[r, "\xi"] & U/\mathbb{G}_{m,B} \arrow[d] \\
\mathrm{Spec}(R') \arrow[r] \arrow[urr, "\xi_{R'}"] & \mathrm{Spec}(R) \arrow[r] & S.
\end{tikzcd} \tag{40}$$

By the valuative criterion in Lemma 3.1, we have the universal closedness of the morphism $U/\mathbb{G}_{m,B} \to S$. $\square$

3.6. Separatedness of $U/\mathbb{G}_{m,B} \to S$

In this Section 3.6, we prove Theorem 2.6.(4) which states that $U/\mathbb{G}_{m,B} \to S$ is separated.

One word about notation: recall that we have $U_L = U \times_S \mathrm{Spec}(L)$ and $U^L = U \times_{U/\mathbb{G}_{m,B}} \mathrm{Spec}(L)$. In this section, we always use upper scripts, such as U^L, U^R, to denote fiber products over $U/\mathbb{G}_{m,B}$; while lower scripts, such as U_L, U_R, $\mathbb{P}^1_L$, denote fiber products over S or B.

We also employ the notation used in Section 3.5.

Suppose the morphism $\xi : \mathrm{Spec}(L) \to U/\mathbb{G}_{m,B}$ in (37) can be extended to a morphism $\xi_0 : \mathrm{Spec}(R) \to U/\mathbb{G}_{m,B}$. The diagram (38) now becomes

$$\begin{tikzcd}
& U^L \arrow[r] \arrow[d, "\alpha_2"'] & \mathrm{Spec}(L) \arrow[l, bend right, "\alpha_1"'] \arrow[r] \arrow[dl, "\eta"'] \arrow[d, "\xi"] & \mathrm{Spec}(R) \arrow[dl, "\xi_0"'] \arrow[d] \\
X & U \arrow[l, hook'] \arrow[r] & U/\mathbb{G}_{m,B} \arrow[r] & S.
\end{tikzcd} \tag{41}$$

Below we prove the separatedness of $U/\mathbb{G}_{m,B} \to S$ by showing that the natural R'-point of $U/\mathbb{G}_{m,B}$ induced by $\xi_0 \in (U/\mathbb{G}_{m,B})(R)$ coincides with the R'-point $\xi_{R'}$ in (40) in Section 3.5. We have the following diagram where every square is Cartesian:

$$\begin{tikzcd}
U^L \arrow[r, "\beta_3"] \arrow[d] & U^R \arrow[r, "\beta_1"] \arrow[d] & U \arrow[d, "\phi"] & U^R \arrow[l, "\beta_1"'] \arrow[d] & U^\kappa \arrow[l, "\beta_2"'] \arrow[d] \\
\mathrm{Spec}(L) \arrow[r] & \mathrm{Spec}(R) \arrow[r, "\xi_0"'] & U/\mathbb{G}_{m,B} & \mathrm{Spec}(R) \arrow[l, "\xi_0"] & \mathrm{Spec}(\kappa). \arrow[l, "i"']
\end{tikzcd} \tag{42}$$

Lemma 3.10. *The fiber products U^κ, U^L, and U^R are all irreducible.*

Proof. Since a geometric fiber of ϕ is a $\mathbb{G}_{m,B}$-orbit, we have the irreducibility of U^κ and U^L. We now show that U^R is irreducible. Since $\mathbb{G}_{m,B}$ is open over B, by [26, Rmk. 2.8.3], we have that $U \to U/\mathbb{G}_{m,B}$ is universally submersive, thus the morphism $U^R \to \mathrm{Spec}(R)$ is submersive. Therefore U^κ is not open. Thus U^L is not closed. Let $\overline{U^L}$ be the closure of U^L inside U^R. If $\overline{U^L} \cap U^\kappa$ is a zero dimensional closed subscheme of U^κ, then the morphism $\overline{U^L} \to \mathrm{Spec}(R)$ violates the upper semicontinuity of dimensions of fibers at the source [19, Ex III.12.7.2]. Therefore we must have $\overline{U^L} \cap U^\kappa$ is one dimensional. Since U^κ is irreducible, we have that $\overline{U^L} \cap U^\kappa = U^\kappa$, and that $U^R = \overline{U^L}$ is irreducible. $\square$

Lemma 3.11. *The set theoretic image of $\beta_1 \circ \beta_2 : U^\kappa \to U$ is contained in the set theoretic image of the restriction of $\pi_7 : Z \to X$ to the closed subscheme $E_n \subset Z$.*

Proof. By Lemma 3.10, we have that for any closed point u of U^κ, $\dim \mathcal{O}_{U^R,u} = 2$. The proof of [30, Lemma 3.35] shows that there exists a finite field extension $L' \supset L$, a discrete valuation ring $R' \subset L'$ dominating R, and $\gamma_{R'} \in U^R(R')$, so that $\gamma_{R'}$ sends the closed points of $\mathrm{Spec}(R')$ to u.

Let $\gamma_{L'}$ be the restriction of $\gamma_{R'}$ to the generic point $\mathrm{Spec}(L') \subset \mathrm{Spec}(R')$. Since $U \to U/\mathbb{G}_{m,B}$ is a geometric quotient, by [16, Def. 0.6.(ii)], we have that the orbit morphism $\mu_\eta : \mathbb{G}_{m,L} \to U$ factors through a surjective morphism $g : \mathbb{G}_{m,L} \to U^L$. Therefore we have a finite field extension $L' \subset L''$ and $\gamma_{L''} \in \mathbb{G}_{m,L}(L'')$ so that $g(\gamma_{L''}) = \gamma_{L'} \in U^L$. Let $R'' \subset L''$ be a valuation ring that dominates $R' \subset L'$. We can identify $\mathbb{G}_{m,L}$ with $\pi_6^{-1}(\mathbb{G}_{m,L}) \subset Z$ since π_6 is an isomorphism over $\mathbb{G}_{m,L} \subset \mathbb{P}^1_R$. Since $Z \to \mathrm{Spec}(R)$ is proper, we can extend $\gamma_{L''} \in \mathbb{G}_{m,L}(L'')$ to $\gamma_{R''} \in Z(R'')$. Let κ'' be the residue field of R''. Let $\gamma_{\kappa''}$ be the restriction of $\gamma_{R''}$ to $\mathrm{Spec}(\kappa'')$.

By construction we have that $\pi_7(\gamma_{L''}) = \beta_1 \circ \beta_2(\gamma_{L'})$. Since X/k is separated, we have that $\pi_7(\gamma_{\kappa''}) = \beta_1 \circ \beta_2(u)$. $\square$

End of Proof of Theorem 2.6.(4). From Lemma 3.11, we see that the set theoretic image of $\beta_1 \circ \beta_2 : U^\kappa \to U$ lies in the set theoretic image of the restriction of $\pi_7 : Z \to X$ to the unique irreducible component $\mathcal{E}$ of E_n that lies between z_j and z_{j+1}. Therefore, the images of all the κ-points of U^κ under $\beta_6 \circ \beta_7$ are in the same $\mathbb{G}_{m,\kappa}$-orbit as $\pi_7(\delta_\kappa)$: recall that δ_κ is defined in Section 3.5 as the closed point $\mathcal{E}$ that is mapped to 1_κ under the natural isomorphism $\mathcal{E} \to \mathbb{P}^1_\kappa$. Furthermore, recall that the image of $\pi_7(\delta_\kappa)$ under the quotient $U \to U/\mathbb{G}_m$ is the closed point of the filling $\xi_{R'} \in U/\mathbb{G}_m(R')$ selected in the end of Section 3.5. Therefore, we have that the composition $\mathrm{Spec}(R') \to \mathrm{Spec}(R) \xrightarrow{\xi_0} U/\mathbb{G}_m$ agrees with $\delta_{\xi'} : \mathrm{Spec}(R') \to U/\mathbb{G}_m$ on both the generic and closed points. Therefore we have a factorization

$$\xi_{R'} : \mathrm{Spec}(R') \to \mathrm{Spec}(R) \xrightarrow{\xi_0} U/\mathbb{G}_m. \tag{43}$$

Since $\xi_{R'}$ is fixed, we have that the lift $\xi_0 \in U/\mathbb{G}_m(R)$ of $\xi \in U/\mathbb{G}_m(\xi)$ is unique, thus we have the separatedness of $U/\mathbb{G}_m \to S$.

The proof of Theorem 2.6.(4), and hence of the Compactification Theorem I 2.6, is now complete. □

4. Projectivity of compactifications in non Abelian Hodge theory

In this section, we apply the compactification/projectivity results of the Projectivity Theorem III 2.8 to first prove Projective Completion Theorems 2.14 (Hodge), 2.18 (de Rham) and 2.20 (Dolbeault). We focus on the Hodge and Dolbeault picture, since Theorem 2.18 is an immediate consequence of the Hodge picture (take the fiber over $t = 1_{\mathbb{A}^1}$ in Theorem 2.14).

4.1. Some more preparatory Lemmata

In this section, we first prove some preparatory Lemmata 4.1, 4.2, 4.3 and 4.5 that are stated in §2.2.

Let J be a field of characteristic $p > 0$. Let C/B be the smooth curve as in (2). Recall that we have fixed the rank r and the degree d for the Hodge, Dolbeault, and de Rham moduli spaces. In the present section, we adopt the following abbreviations: Let A be the Hithcin base $A(C/B)$ for the curve C/B. Let $A^{(B)}$ be the relative Frobenius twist of A. Let A' be the Hitchin base $A(C^{(B)}/B)$ for the curve $C^{(B)}/B$.

Lemma 4.1 *($A^{(B)} \cong A'$). There exists an isomorphism of B-schemes $A^{(B)} \cong A'$.*

Proof. To find an isomorphism between $A^{(B)}$ and A' is equivalent to find a natural isomorphism between sheaves of $\mathcal{O}_B$-algebras:

$$Sym^\bullet((\pi_*\omega_{X/B})^\vee) \otimes_{\mathcal{O}_B, fr_B^\#} \mathcal{O}_B \cong Sym^\bullet((\pi_*^{(B)}\omega_{X^{(B)}/B})^\vee). \tag{44}$$

Since X/B is smooth and of relative dimension 1, working etale locally over B, we can assume that B is an affine scheme $\mathrm{Spec}(R)$, and $X = \mathrm{Spec}(R[x])$. The coherent sheaf $\omega_{X/B}$ (resp. $\omega_{X^{(B)}/B}$) is the rank 1 free $R[x]$-module (resp. $R[x\otimes 1]$-module) with a generator dx (resp. $d(x \otimes 1)$). Let y (resp. $y \otimes 1$) be the element in $Hom_R(R[x], R)$ (resp. $Hom_R(R[x \otimes 1], R)$) that sends x (resp. $x \otimes 1$) to $1 \in R$. The left hand side of (44) corresponds to the R-algebra $R[y, \partial_x] \otimes_{R, fr_R} R$, while the right hand side of (44) corresponds to the R-algebra $R[y \otimes 1, \partial_{x\otimes 1}]$. The assignment $f\partial_x \otimes 1 \mapsto (f \otimes 1)\partial_{x\otimes 1}$

induces an isomorphism of R-algebras from the left hand side to the right hand side of (44). $\square$

Remark 4.2 *(Fr_A "=" σ_B^*).* Assuming the notation in Lemma 4.1 and its proof, we see that the comorphisms of the B-morphisms $A \to A^{(B)} \to A'$ are determined by the assignments $(f \otimes 1)\partial_{x\otimes 1} \mapsto f\partial_x \otimes 1 \mapsto (f\partial_x)^p \in R[y, \partial_x]$. We can then derive another description of the compositum $F : A \to A'$:

A B-point of A is a linear combination of terms of the form $r_i x^j (dx)^k$ with $r_i \in R$. We have that

$$((y \otimes 1)^j \partial^k_{x\otimes 1})\Big(F(B)\big(r_i x^j (dx)^k\big)\Big) = \Big((y^j \partial^k_x)\big(r_i x^j (dx)^k\big)\Big)^p = r_i^p.$$

Therefore we see that $F(B)$ sends dx to $dx \otimes 1$, x to $x \otimes 1$, and r_i to r_i^p. Therefore, the function $F(B) : A(B) \to A'(B)$ is induced by the pull back

$$\begin{aligned} & H^0(B \times_B C, pr_C^* \omega_{C/B}) && (= H^0(C, \omega_{C/B})) \\ \xrightarrow{(fr_B^*, \sigma_B^*)} & H^0(B \times_B C^{(B)}, pr^*_{C^{(B)}} \omega_{C^{(B)}/B}) && (= H^0(C^{(B)}, \omega_{C^{(B)}/B})). \end{aligned}$$

Similarly, one can show that, for any B-scheme T, the morphism $F(T) : A(T) \to A'(T)$ is induced by the pull back $(\sigma_B, fr_T)^*$.

Note that by [29, §2.3], we have the following commutative diagram:

$$\begin{tikzcd} C^{(B)} \times_B T \arrow[r, "\simeq"'] \arrow[rr, bend left, "(\sigma_B, fr_T)"] \arrow[dr, "pr_T"'] & (C \times_B T)^{(T)} \arrow[r, "\sigma_T"'] \arrow[d] & C \times_B T \arrow[d] \\ & T \arrow[r, "fr_T"'] & T. \end{tikzcd} \tag{45}$$

Therefore, up to a natural identification $C^{(B)} \times_B T \cong (C \times_B T)^{(T)}$, the morphism $F(T)$ is induced by σ_T^*.

Lemma 4.3 *(**Factorization of Hodge-Hitchin Morphism over** $0_{A'}$). Let A^p be the B-scheme that is the total space of the locally free $\mathcal{O}_B$-module $\bigoplus_{i=1}^r \pi_* \omega_{X/B}^{\otimes ip}$. There exists the following commutative diagram:*

$$\begin{tikzcd} M_{Hod}(C/B)_{0_{A'}} \arrow[r, "h_{Hod, 0_{A'}}"] & A' \arrow[r, "Fr^*_{C/B}"] & A^p \\ M_{Dol}(C/B) \arrow[r, "h_{Dol}"'] \arrow[u] & A. \arrow[u, "\sigma_B^* \text{“=”} Fr_A"'] \arrow[ur, "fr_C^*"'] & \end{tikzcd} \tag{46}$$

Proof. Since the p-curvature of a Higgs bundle ϕ is ϕ^p, we have that the outer 5-gon in (46) is commutative. Since $fr_C = \sigma_B \circ Fr_{C/B}$, by Remark 4.2, we see that the triangle in (46) is commutative. Furthermore, the morphism $Fr^*_{C/B}$ is a monomorphism. Therefore, the bottom left square of (46) is commutative. $\square$

Remark 4.4. The commutativity of diagram (46) shows that the isomorphism $A^{(B)} \cong A'$ is $\mathbb{G}_{m,B}$-equivariant, since all the other arrows involved in the left square of (46) are $\mathbb{G}_{m,B}$-equivariant.

Lemma 4.5 *(**Trivialization of Hodge-Hitchin Morphism over** $\mathbb{G}_{m,B}$). The natural B-morphism $M_{dR}(C/B) \to M_{Hod}(C/B) \times_{\mathbb{A}^1_B} 1_B$ is an isomorphism. There exists a natural isomorphism of B-schemes $M_{dR}(C/B) \times_B \mathbb{G}_{m,B} \cong M_{Hod}(C/B) \times_{\mathbb{A}^1_B} \mathbb{G}_{m,B}$. Furthermore, we have the following commutative diagram:*

$$\begin{tikzcd}
M_{Hod}(C) \times_{\mathbb{A}^1} \mathbb{G}_{m,B} \arrow[d, "h_{Hod,\mathbb{G}_{m,B}}"'] & M_{dR}(C) \times_B \mathbb{G}_{m,B} \arrow[l, "\simeq"'] \arrow[d, "h_{dR} \times 1_{\mathbb{G}_{m,B}}"] \\
A' \times \mathbb{G}_{m,B} & A' \times \mathbb{G}_{m,B} \arrow[l, "\simeq"']
\end{tikzcd} \tag{47}$$

Proof. Since M_{Hod} is uniformly corepresenting, we have that the fiber product $M_{Hod} \times_{\mathbb{A}^1} \mathbb{G}_{m,B}$ is corepresenting the functor of t-connections with invertible t. Therefore, the morphism $((E, \nabla), t) \mapsto (E, t\nabla))$ defines an isomorphism between the functors that are corepresented by $M_{dR} \times_B \mathbb{G}_{m,B}$ and $M_{Hod} \times_{\mathbb{A}^1} \mathbb{G}_{m,B}$, thus an isomorphism between the corepresenting schemes. Since f is also an isomorphism of $\mathbb{G}_{m,B}$-schemes, we have an isomorphism $M_{dR} \cong M_{Hod} \times_{\mathbb{A}^1} 1_B$. When J is a field of characteristic $p > 0$, the bottom isomorphism in (47) is given by $(a_i, t) \mapsto (t^{ip} a_i, t)$, with $a_i \in \mathbb{A}'_i$ (recall that $\mathbb{A}'_i$ is the direct factor of $A(C^{(B)}/B)$ corresponding to the locally free sheaf $\pi^{(B)}_* \omega^{\otimes i}_{X^{(B)}/B}$). $\square$

4.2. Proof of Theorems 2.13 and 2.14

Proof of Theorem 2.13. We have the basic $\mathbb{G}_{m,B}$-equivariant diagram with Cartesian square (where the subscript B is dropped) from [15, §3.1]. This construction already appears in [20, Proof of Lemma 6.1].

$$\begin{tikzcd}
Z := M_{Hod} \times_{\mathbb{A}^1} \mathbb{A}^2 \arrow[r] \arrow[d] \arrow[dd, bend right=60, "\tau'"'] & M_{Hod} \arrow[d, "\tau"] & & \\
\mathbb{A}^2_{x,y} \arrow[r] \arrow[d] & \mathbb{A}^1_\lambda & (x, y) \arrow[r, mapsto] \arrow[d, mapsto] & \lambda = xy \\
S := \mathbb{A}^1_x & & x, &
\end{tikzcd} \tag{48}$$

where the $\mathbb{G}_{m,B}$ action on $\mathbb{A}^2_{x,y}$ is defined by setting $t(x,y) := (x, ty)$, the $\mathbb{G}_{m,B}$ action on $\mathbb{A}^1_\lambda$ is the usual dilation $t \cdot \lambda := t\lambda$, and the $\mathbb{G}_{m,B}$ action on $\mathbb{A}^1_x$ is trivial.

We would like to apply the Compactification Theorem II 2.7 to Z/S in (48). Below we show that the assumptions in Theorem (2.7), i.e. that zero limits of Zariski points exist in Z, and that the fixed point locus in Z is proper over S, are satisfied:

A Zariski point $z \in Z(k)$, where k is a field, can be represented by a pair $((E,\nabla),(x,y))$ on C_k, where (E,∇) is semistable as a vector bundle with an xy-connection. The $\mathbb{G}_{m,k}$-orbit of z is then represented by $((E,t\nabla),(x,ty))$. We can naturally extend this $\mathbb{G}_{m,k}$-orbit to obtain an $\mathbb{A}^1_k$-family with the element over $0_k \in \mathbb{A}^1_k$ being $((E,\phi),(x,0))$, where (E,ϕ) is a possibly non-stable Higgs bundle. By Langer's Langton-type result [27, Th. 5.1], we can change the 0_k-fiber (E,ϕ) to a semistable (E',ϕ') so that we obtain a morphism $\mathbb{A}^1_k \to Z$ which maps $t \in \mathbb{G}_{m,k}(k)$ to $((E,t\nabla),(x,ty))$ and maps 0_k to (E',ϕ'). Therefore, we have that Z has all its zero limits.

The $\mathbb{G}_{m,B}$ fixed locus on M_{Hod} is contained in the fiber $M_{Hod,0_{\mathbb{A}^1_\lambda}}$ of τ over the origin $0_{\mathbb{A}^1_\lambda} \in \mathbb{A}^1_\lambda$. Since the natural morphism $M_{Dol} \to M_{Hod,0_{\mathbb{A}^1_\lambda}}$ is $\mathbb{G}_{m,B}$-equivariant and bijective on geometric points, we have that the set underlying $\mathbb{G}_{m,B}$ fixed locus is naturally identified with the $\mathbb{G}_{m,B}$ fixed locus of M_{Dol} inside $h^{-1}_{Dol}(o_{A'})$.

We now show that $h^{-1}_{Dol}(o_{A'})$ is proper over B: Using the valuative criterion for properness, we are reduced to the case where B is a discrete valuation ring, but then the Langton-type argument in the proof of [14, Thm. I.3] goes through and gives the properness of $h^{-1}_{Dol}(o_{A'})$ over B. Since the B-morphism $M_{Dol} \to M_{Hod,0}$ is bijective on geometric points, using the valuative criterion Lemma 3.1 (taking L to be algebraic closure of K so that we can lift the L point from $M_{Hod,0_{\mathbb{A}^1}}$ to M_{Dol}), we have that the $\mathbb{G}_{m,B}$-fixed locus of M_{Hod} is also proper over B.

The following can then be verified:

(1) The $\mathbb{G}_{m,B}$ fixed point set in Z is proper over S;

(2) The complement U in Z of the set of points in Z admitting infinity limits, is Z minus the x-axis times the closed subset of M_{Hod} that is universally homeomorphic to the proper fiber $h^{-1}_{Dol}(o_{A'})$;

(3) The open subvariety $T \subseteq Z$ obtained by removing the preimage of the origin via the projection onto the y-axis, endowed with the projection to this puncture y-axis is $\mathbb{G}_{m,B}$-equivariantly isomorphic to $M_{Hod} \times \mathbb{G}_{m,B}$ (cf. (47)).

Apply the Projectivity Theorem III 2.8 to this situation. The Projective Completion of $\tau : M_{Hod} \to \mathbb{A}^1$ Theorem 2.13 follows once we set $\overline{M_{Hod}} := U/\mathbb{G}_{m,B}$ etc.

To finish the proof, we simply need to observe that:

(i) M_{Hod} admits a natural open immersion into $\overline{M_{Hod}}$ by property (3) above.

(ii) $U/\mathbb{G}_{m,B}$ admits a natural $\mathbb{G}_{m,B}$ action compatible with the open immersion (i); this action is already present on Z and on U by letting $\mathbb{G}_{m,B}$ act by the standard weight one dilation $t \cdot x := tx$ on the x coordinate.

(iii) We also need to endow $S := \mathbb{A}^1_x$ (which originally had the trivial action, so we could arrive to $U/\mathbb{G}_{m,B} \to S$ via the Compactification Theorem II 2.7) with the standard weight one dilation action, so that, now, $\overline{\tau}$, and in fact the whole diagram (8), is $\mathbb{G}_{m,B}$-equivariant. The proof of Theorem 2.13 is thus completed. $\square$

Proof of Theorem 2.14. We would like to apply the Compactification Theorem III 2.8. We define the Z, Z', and S in Theorem 2.8 in the following way:

We augment the diagram (48) by inserting the Hodge-Hitchin morphism. We obtain the $\mathbb{G}_{m,B}$-equivariant commutative diagram with Cartesian squares

$$\begin{tikzcd}
& Z := M_{Hod} \times_{\mathbb{A}^1} \mathbb{A}^2 \arrow[r] \arrow[d, "h'_{Hod}"] \arrow[ddl, bend right, "\tau'"'] & M_{Hod} \arrow[d, "h_{Hod}"] \arrow[dd, bend left, "\tau"] \\
& Z' := A' \times_B \mathbb{A}^2_{x,y} \arrow[r] \arrow[d] & A' \times_B \mathbb{A}^1_{\lambda} \arrow[d] \\
S := \mathbb{A}^1_x & \mathbb{A}^2_{x,y} \arrow[l] \arrow[r] & \mathbb{A}^1_{\lambda}.
\end{tikzcd} \tag{49}$$

To check that the assumptions of Theorem 2.8, we note that, firstly, the assumptions in Theorem 2.8 that are only about Z and U are checked in the proof of Theorem 2.13; secondly, other assumptions involving Z' and U' are easily checked to be satisfied. The existence of the commutative diagram (9), the item (1) of Theorem 2.14, and the projectivity of $\overline{h_{Hod}}$ and $\overline{\tau_{Hod}}$ then follows from the application of Theorem 2.8.

Since $\overline{h_{Hod}}$ maps the boundary of $\overline{M_{Hod}}$ to the boundary $\overline{A'} \times \mathbb{A}^1_B$, we have that the Hodge-Hitchin morphism h_{Hod} is also projective. Thus we have the item (2) of Theorem 2.8.

By inspecting the construction of Z', it is clear that we obtain the weighted projective space $\mathbb{P}(1, 1 \cdot p, 2p, \ldots, rp)$. This latter coincides with $\mathbb{P}(1, 1, 2, \ldots, r)$ in view of the fact that we can replace ip by i; see Remark 2.15. The proof of Theorem 2.14 is thus completed. $\square$

4.3. Proof of Theorems 2.19 and 2.20

Proof of Theorem 2.19. We would like to apply Compactification Theorem III 2.8 as in the proof of Theorem 2.14 in §4.2.

To obtain a similar diagram as (49), we substitute the right column of (49) by the morphisms $M_{Dol} \xrightarrow{h'_{Dol}} A \times_B \mathbb{A}^1 \xrightarrow{pr} \mathbb{A}^1$, where h'_{Dol} denotes the Hitchin morphism $h_{Dol} : M_{Dol} \to A$ followed by the closed embedding $A = A \times \{0_{\mathbb{A}^1}\} \to A \times \mathbb{A}^1$. We can then add the corresponding analogue of the left column in (49) by the construction in (48). Then the arguments in §4.2 applies, *mutatis mutandis*, and finishes the proof of all the statements in Theorem 2.19 about the left half of the diagram (11).

For the remaining statements in Theorem 2.19, we need two variations of the diagram (48): For the first variation, we first replace the right column of (48) by the constant B-

morphism $M_{Dol} \to \mathbb{A}^1_B$ that sends M_{Dol} to $0_B \subset \mathbb{A}^1_B$, and then construct the left column as in (48)– this gives us the compactification of $\overline{M_{Hod}}$ above; For the second variation, we take the base change of the whole diagram (48) via the inclusion $0_\lambda \hookrightarrow \mathbb{A}^1_\lambda$– this gives us the compactification $\overline{M_{Hod,0_B}}$ and shows that the fiber $(\overline{M_{Hod}})_{0_B}$ of the compactification of M_{Hod} constructed in Theorem 2.13 is the compactification of $M_{Hod,0_B}$. The natural morphism $M_{Dol} \to M_{Hod,0_B}$ then induces a morphism of the two variations of diagram (48), thus a morphism $\overline{M_{Dol}} \to \overline{M_{Hod,0_B}}$. The remaining statements in Theorem 2.19 can then be checked routinely. $\square$

Proof of Theorem 2.20. We assume the notation in the proof of Theorem 2.19. Consider the commutative diagram of $\mathbb{G}_{m,B}$-equivariant morphisms

$$\begin{tikzcd} M_{Dol} \arrow[rr] \arrow[d, "h'_{Dol}"] & & M_{Hod} \arrow[d, "h_{Hod}"] \\ A \times_B \mathbb{A}^1 \arrow[rr, "Fr_A \times Id_{\mathbb{A}^1}"] \arrow[dr, "pr"'] & & A' \times_B \mathbb{A}^1 \arrow[dl, "pr"] \\ & \mathbb{A}^1. & \end{tikzcd} \tag{50}$$

We repeat essentially verbatim the arguments in §4.2, by applying the construction (48) and augmenting it using (50), the same way we used (49). Of course, we need the evident "multiple morphisms" version of the Compactification Theorem II 2.7 and of the Projectivity Theorem III 2.8. The items (1)-(3) of Theorem 2.20 then follow. (Let us remark that, in view of the factorization $M_{Dol} \to M_{Hod,0_{\mathbb{A}^1}} \to A'$, the projectivity of $M_{Dol} \to M_{Hod,0_{\mathbb{A}^1}}$ follows immediately from the projectivity of the compositum $M_{Dol} \to A \to A'$.)

For the item (4) of Theorem 2.20, note that $c : M_{Dol} \to M_{Hod,0_{\mathbb{A}^1}}$ is a universally closed bijection. If the rank r and degree d are coprime, then the morphism c is an isomorphism by the universal corepresentability property of the Hodge moduli space; see Remark 2.12. We are done if we can show that c is indeed a universal bijection, as universally closed universal bijections are universal homeomorphisms. Given any Zariski point x of M_{Dol}, we would like to show that the field extension $\kappa(x) \supset \kappa(c(x))$ is purely inseparable. By taking the closure of x and $c(x)$, and shrinking $\overline{c(x)}$ if necessary, we may assume that $\overline{x}$ and $\overline{c(x)}$ are both normal, and $c|_{\overline{x}}$ is finite. Let $\tilde{x}$ be the normalization of $\overline{x}$ in the separable closure of $\kappa(c(x))$ inside $\kappa(x)$. We have that $\tilde{x} \to \overline{c(x)}$ is generically etale and injective. Since J is algebraically closed, and here is the only place where we use this assumption on J, we have that $\tilde{x} \to \overline{c(x)}$ is an isomorphism over an open and dense subset of $\overline{c(x)}$. Therefore, we have that $\kappa(\tilde{x}) = \kappa(c(x))$, thus the field extension $\kappa(x) \supset \kappa(c(x))$ is purely inseparable. We have thus proved item (4).

For the item (5) of Theorem 2.20, note that by construction, we obtain the morphism

$$\overline{A} = \mathbb{P}(1,1,2,\ldots,p) \longrightarrow \overline{A'} = \mathbb{P}(1,p,2p,\ldots,rp) = P(1,1,2,\ldots,r) = \overline{A}. \tag{51}$$

Note that the Frobenius twist of $\overline{A}$ is $\mathbb{P}(p,p,2p,...,rp) = \mathbb{P}(1,1,2,...,p) = \overline{A} = \overline{A'}$. Since the restriction of $\overline{A} \to \overline{A'}$ to A is the relative Frobenius Fr_A, we have that the morphism (51) $\overline{A} \to \overline{A'}$ is also the relative Frobenius $Fr_{\overline{A}}$ for $\mathbb{P}(1,1,2,\ldots,r)$.

The compactification and M_{Dol} Theorem 2.20 follows. $\square$

5. Appendix: smooth moduli and specialization

5.1. Introduction to the appendix

The paper [7] is devoted to develop a formalism for the specialization morphism, when it exists, as a perverse Leray filtered morphism for a family of morphisms $f : X \to Y$ over a base curve S; the ground field is the one of the complex numbers. While §1,2,3 of [7] are rather general, §4 of [7] is devoted to applying the formalism when the morphism f is the Hitchin morphism $M_{Dol}(C/S) \to A(C/S) \to S$ associated with a smooth curve (§2.2) C/S. The main result is [7, Tm 4.4.2], to the effect that the specialization morphisms for this family are defined and are filtered isomorphisms. Another relevant result is [7, Lm 4.3.3], to the effect that $\phi(v_*\mathbb{Q}_{M_{Dol}}) = 0$ for the vanishing cycle functor applied to the direct image complex with respect to the structural morphism $v : M_{Dol}(C/S) \to S$. Note that neither statement is a priori clear, since the morphism v is not proper.

In the paper [8], we use the generalization of [7, Tm. 4.4.2, Lm 4.3.3] to the cases where the base is a complete strictly Henselian DVR S, and the morphisms f are: 1) the Hodge-Hitchin morphism (5) $M_{Hod}(C/k) \to \mathbb{A}(C^{(1)}) \times A^1_k \to \mathbb{A}^1_k$, after base change to the appropriate local ring S at the origin of $\mathbb{A}^1_k$; 2) the Hitchin morphism (6) $M_{Dol}(C/S) \to A(C/S) \to S$. These moduli space are with respect to certain coprimality conditions on rank, degree and characteristic of the ground field. These conditions turn out to imply the smoothness of these moduli spaces and that they universally corepresent the appropriate functors (so that taking the fibers over S, one gets the expected moduli space). The desired generalization of these results is not a simple matter of routine and has as starting point the compactification results stated in §2.2 and proved in §4.2.

The purpose of this appendix is to tie in the compactification results of this paper with [7, §4] by providing the necessary background so that we can prove [7, Tm. 4.4.2, Lm. 4.3.3] for the Hodge and Dolbeault moduli spaces for smooth curves C/S (2.2) over a complete strictly Henselian DVR S as in the previous paragraph, so that we can use these results in [8].

Brief summary of [7, §4]. [7, §4] uses the compactification constructed in [6, Tm. 3.1.1 and (14)]; this construction uses the same kind of quotient by $\mathbb{G}_m$ technique used in this paper. The outcome of the construction is summarized in the diagrams [7, (70), (72)]: the compactification of the moduli space is denoted by an open immersion $X^o \subseteq X$

with boundary Z (i.e. a triple (Z, X, X^o)); the compactification is obtained by taking the quotient by $\mathbb{G}_m$ of a suitable triple $(\mathscr{Z}, \mathscr{X}, \mathscr{X}^o)$. [7, §4] uses in an essential way, the topological local triviality, due to C. Simpson, of the Dolbeault moduli spaces over the base S (all over the complex numbers). This implies the same kind of local triviality of the triple $(\mathscr{Z}, \mathscr{X}, \mathscr{X}^o)$. In turn, this implies the vanishing $\phi(\mathbb{Q}_{\mathscr{X}}) = \phi(\mathbb{Q}_{\mathscr{Z}}) = 0$ of the vanishing cycles before taking the quotient by $\mathbb{G}_m$. Due to the local product structure ([7, Lm. 4.3.1]), one also has, before taking the quotient, that $a^!\overline{\mathbb{Q}}_{\ell\mathscr{X}} = \overline{\mathbb{Q}}_{\ell\mathscr{Z}}[-2]$, where $a : \mathscr{Z} \to \mathscr{X}$ is the closed embedding. The key point in proving [7, Tm.4.4.2, Lm. 4.3.3] is to descend, along the quotient by $\mathbb{G}_m$, the vanishings and identities above from the triple $(\mathscr{Z}, \mathscr{X}, \mathscr{X}^o)$ to the triple (Z, X, X^o).

Plan to fulfill the purpose. In order to achieve the desired purpose stated above, the plan is to follow the path traced in [7, §4] over the complex numbers and make the necessary adjustments along the way when working over the DVR S.

The suitable $\overline{\mathbb{Q}}_\ell$-adic formalism in §5.2. First of all, we need a suitable formalism of $\overline{\mathbb{Q}}_\ell$-adic constructible sheaves on separated schemes of finite type over the complete strictly Henselian DVR S. This is the content of §5.2, where we collect some results in the literature to provide a linear exposition of the eight functor formalism and of perverse sheaves for $\overline{\mathbb{Q}}_\ell$-adic constructible complexes on separated schemes of finite type over an excellent DVR. With the formalism of §5.2 at our disposal, we recover virtually the whole of the machinery in [7, §1,2,3], and we are ready to tackle the purpose of this appendix.

The compactifications we use. We use the compactifications of Hodge and Dolbeault moduli given in §2.2 by Theorems 2.14 and 2.20. The key construction is summarized by diagrams (48) and (49). Warning on notation concerning case of the Hodge moduli space: i) what has been denoted by $\mathscr{X}$ above in the brief summary is a suitable open subset U of what is denoted Z in (49) (cf. the proof of Theorem 2.13 in §4.2); ii) what is denoted by $\mathscr{Z}$ above, is the closed subset of $U = \mathscr{X}$ given by the preimage of the x-axis with the fiber of the Hitchin morphism over the origin (nipotent cone N_{Dol}) removed from every copy of M_{Dol} over the points of the x-axis; then $\mathscr{Z}$ is isomorphic to the product (x-axis)$\times(M_{Dol} \setminus N_{Dol})$. Despite the spaces being singular, the closed embedding $\mathscr{Z} \subseteq \mathscr{X}$ is regular of codimension one.

In general, the Hodge and Dolbeault moduli spaces are not regular. We leave the issues of smoothness over the appropriate base, and of universal corepresentability for the Hodge and Dolbeault moduli spaces to [8] (they follow from suitable coprimality assumptions). Next, we tackle all the remaining issues that arise in connection to generalizing [7, Th. 4.4.2, Lm. 4.3.3] from $\mathbb{C}$ to a complete strictly Henselian DVR S.

List of remaining technical issues in §5.3. At this juncture, a close inspection of [7] reveals that the only issues that arise when trying to fulfill the purpose of this appendix, i.e. generalize [7, Lm. 4.3.3 and Tm. 4.4.2] to these compactifications discussed above over the DVR S, are the aforementioned smoothness and universal corepresentability (dealt with in [8]), and a small list of technical facts that need to be suitably generalized when replacing the base ring $\mathbb{C}$ with an algebraically closed field, or with a complete strictly Henselian DVR. Dealing with this list, is the content of §5.3.

5.2. Rectified perverse t-structure over a DVR

We collect and complement references in the literature, so that one can work with nearby/vanishing cycles in the context of $\overline{\mathbb{Q}}_\ell$-adic coefficients and middle perversity t-structures. The key ingredient is O. Gabber rectified middle perversity t-structure for schemes of finite type over a DVR.

The trait. Let (S, s, η) be trait, i.e. the spectrum of a DVR (discrete valuation ring), with closed point $i : s \to S$, and with generic open point $j : \eta \to S$. With the exception of (52), we work with schemes $f : X \to S$ that are separated and of finite type over S and with S-morphisms that are separated and of finite type. The special closed fiber is denoted X_s and the generic open fiber X_η. Fix a prime number ℓ that is invertible in S.

The constructible $\overline{\mathbb{Q}}_\ell$-adic derived category. The trait S is Noetherian, regular and of dimension one. In particular, we have access to the finiteness results in [38, Th. Finitude]. Let X/S be as above. Let $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ be the $\overline{\mathbb{Q}}_\ell$-constructible derived category, whose objects we call (constructible) complexes; see [11, Thm. 6.3]; see also [14, §5]. It is endowed with a natural t-structure, with heart the abelian category of $\overline{\mathbb{Q}}_\ell$-constructible sheaves on X.

The formalism of functors. As X/S varies, with S fixed, the categories $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ enjoy the usual formalism of the eight ("derived") functors

$$(f_!, f^!), (f^*, f_*), (\otimes, \mathrm{Hom}), \psi, \phi,$$

with the usual adjunction relations among them –in each parenthesis (A, B) above, A is left adjoint to B–, as well as duality exchanges –such as $Df_! = f_*D, Df^! = f^*D, D\psi[-1] = \psi[-1]D$ (for the last one, use [23, Thm.4.6], and $\psi = \psi_\eta \circ j^*$). These functors are exact (additive, commute with translations, preserve distinguished triangles). See [11, Th. 6.3], for a partial list of the properties concerning $f_!, f^!, f^*, f_*, \otimes, \mathrm{Hom}$. The functors D, ψ and ϕ are discussed below.

Duality. The duality functor D used above is introduced as follows. We call the constructible complex $K_S := \overline{\mathbb{Q}}_{\ell S}[2](1)$ the dualizing sheaf of S; see [11, Tm. 6.3.(iii)]. The object $K_{X/S} := f^! K_S$ is a dualizing sheaf on X relative to S. We denote by $D := \mathrm{Hom}(-, K_{X/S}) : D^b_c(X, \overline{\mathbb{Q}}_\ell) \to D^b_c(X, \overline{\mathbb{Q}}_\ell)$ the corresponding dualizing contravariant functor. Note that $K_{s/S} = i^! K_S = K_{s/s} = \overline{\mathbb{Q}}_{\ell s}$, i.e. the dualizing sheaf of s relative to S coincides with the usual dualizing sheaf of s. On the other hand, $K_{\eta/S} = j^! K_S = \overline{\mathbb{Q}}_{\ell\eta}[2](1) \neq \overline{\mathbb{Q}}_{\ell\eta} = K_{\eta/\eta}$.

Nearby and vanishing cycles functors. For the nearby and vanishing cycles functors ψ and ϕ see [39, 7.1 I, 7.2 XIII], as well as [23,24], and references therein. Note that what we denote by ϕ here, is denoted by $\phi[-1]$ in [23,24]; in particular, the usual distinguished triangle of functors appears here as (54) $i^* \to \psi \to \phi[1] \rightsquigarrow$.

When using the functors ψ, ϕ we assume in addition that the trait $S = S^h$ is Henselian. Choose a separable closure of the residue field. Form the associate strict Henselianization $S_{(\overline{s})}$ of S at s. Choose a separable closure of the fraction field of the strict Henselianization. After base change, we obtain natural morphisms of S-schemes (ϵ and $\overline{j}$ are not of finite type)

$$X_{\overline{s}} \xrightarrow{\overline{i}} X_{S_{(\overline{s})}} \xleftarrow{\overline{j}} X_{\overline{\eta}} \xrightarrow{\epsilon} X_\eta \xrightarrow{j} X. \tag{52}$$

We define the nearby cycles functor by setting

$$\psi := \overline{i}^* \overline{j}_* \epsilon^* j^* : D^b_c(X, \overline{\mathbb{Q}}_\ell), \longrightarrow D^b_c(X_{\overline{s}}, \overline{\mathbb{Q}}_\ell). \tag{53}$$

The constructibility assertion here is from [38, Th. Finitude, Tm. 3.2] and [23, p.45 top] (this is what is needed in [11] to land in the constructible derived category). We also have the more classical nearby cycles functor $\psi_\eta : D^b_c(X_\eta, \overline{\mathbb{Q}}_\ell) \to D^b_c(X_{\overline{s}}, \overline{\mathbb{Q}}_\ell)$, obtained by setting $\psi_\eta := \overline{i}^* \overline{j}_* \epsilon^*$. Clearly, $\psi = \psi_\eta \circ j^*$, i.e. $\psi(F)$ depends only on the restriction $j^* F$. Let $\nu : X_{S_{(\overline{s})}} \to X$ be the natural projection morphism. By adjunction, we have a natural morphism $\overline{i}^* \nu^* \to \psi$, which we simply denote by $i^* \to \psi$ (this is literally correct if S is strictly Henselian). The cone of this morphism is in fact functorial (cf. [39, 7.2 XIII, (1.4.2.2, 2.1.2.4)], and we denote it by $\phi[1]$. We have a distinguished triangle of functors

$$i^* \to \psi \to \phi[1] \rightsquigarrow . \tag{54}$$

For a partial list of the properties concerning ψ and ϕ, see [7, §2.1]. See also [23, §4] (in particular, see Th. 4.7 for the compatibility with cup products).

The rectified middle perversity t-structure. We go back to the case where S is a trait (not necessarily Henselian). For X/S separated and of finite type, the category $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ is endowed with O. Gabber's "rectified" middle-perversity t-structure [23,24, §4, §2] which is defined by setting

$$^pD^{\leq 0}(X, \overline{\mathbb{Q}}_\ell) = \{K \mid i^* K \in {}^pD^{\leq 0}(X_s),\ j^* K \in {}^pD^{\leq -1}(X_\eta)\}, \tag{55}$$

$$^pD^{\geq 0}(X, \overline{\mathbb{Q}}_\ell) = \{K \mid i^! K \in {}^pD^{\geq 0}(X_s),\ j^* K \in {}^pD^{\geq -1}(X_\eta)\}.$$

If X/S is obtained from a separated scheme Z of finite type over a smooth curve C over a field, by localizing at a closed point on the curve C, then the usual middle perversity t-structure on Z induces the rectified one in (55).

Note that $i_* \overline{\mathbb{Q}}_{\ell s}$, $\overline{\mathbb{Q}}_{\ell S}[1] = R^0 j_* \overline{\mathbb{Q}}_{\ell \eta}[1]$, $j_* \overline{\mathbb{Q}}_{\ell \eta}[1]$ and $j_! \overline{\mathbb{Q}}_{\ell \eta}[1]$ are rectified perverse.

Self-duality. By using the definition, and the fact that the middle perversity t-structure on a variety over a field is self-dual for the usual relative dualizing complexes over the field, it is easy to verify that the rectified perverse t-structure is self-dual for the relative dualizing complex $K_{X/S}$, i.e. the duality functor D exchanges ${}^{p}D^{\leq 0}(X, \overline{\mathbb{Q}}_\ell)$ and ${}^{p}D^{\geq 0}(X, \overline{\mathbb{Q}}_\ell)$.

t-exactness. The functors j_* and $j_!$ and $(j^*[-1] = j^![-1]$, $\psi[-1]$ and ϕ are t-exact ([23, §4]). If $f : X \to Y$ is an affine S-morphism, and S is a Henselian, then f_* is right t-exact for the rectified perverse t-structure ([24, Th. 2.4, due to O. Gabber]. Let $f : X \to Y$ be a morphism. Let $d \geq 0$ be an integer such that every geometric fiber of f has dimension at most d. Then we have the following "inequalities" for the rectified t-structure: $f_! : {}^{p}D^{\leq 0}(X) \to {}^{p}D^{\leq d}(Y)$, $f^! : {}^{p}D^{\geq 0}(Y) \to {}^{p}D^{\geq -d}(X)$, $f^* : {}^{p}D^{\leq 0}(Y) \to {}^{p}D^{\leq d}(X)$, $f_* : {}^{p}D^{\geq 0}(X) \to {}^{p}D^{\geq -d}(Y)$; see [3, 4.2.4] for the case of a field, that can be used to bootstrap the proof over S by using (55).

We also have the analogues of [3, 4.1.10-11-12, Prop. 4.2.5, 4.2.6], which can be proved in the same way.

The category of rectified perverse sheaves. The category of rectified perverse sheaves on X is Abelian, Noetherian, self-dual, Artinian, and every simple object is an intermediate extension of a simple object ([3, §4.3]) from either the central, or the generic fiber (for this last item, see [23, p.49, (c)]).

5.3. Compactification and specialization

We refer to §5.1. Let $C/B = J$ be a smooth curve as in §2.2. The technical issues we need to address are the following:

(a) The construction, of a suitable natural completions of: For $J = k$ an algebraically closed field, $M_{Hod}(C/k)/\mathbb{A}^1_k$ and of the associated Hodge-Hitchin morphism $M_{Hod}(C/k) \to A(C^{(1)}) \times \mathbb{A}^1_k$, for $J = S$ a complete strictly Henselian DVR, of $M_{Dol}(C/B)/B$ and of the associated Hitchin morphism $M_{Dol}(C/B) \to A(C/B)$). This way, diagram [7, (70) and (72)] and their properties are in place (they are Cartesian up to nilpotents, and this creates no problems when working with the étale topology).
(b) Taking the quotient by a possibly non-reduced flat finite group subscheme of $\mathbb{G}_{m,B}$ in the proof of [7, Lemma 4.1.1].
(c) Being able to factor the $\mathbb{G}_m$ quotient into a quotient as in (b), and a free quotient, as in [7, Lemma 4.1.1].
(d) The use of Luna slice Theorem in the proof of [7, Lm. 4.1.1] in the context of a $\mathbb{G}_{m,B}$ action with trivial stabilizers on an affine variety;
(e) The use of Lemma [7, Lm. 4.1.4], i.e. the assertion that if $p : A \to C$ is the quotient by a finite flat group scheme G over B (with some extra assumptions to be listed in Lemma 5.3), then $\overline{\mathbb{Q}}_{\ell C}$ is a direct summand of $p_* \overline{\mathbb{Q}}_{\ell A}$.
(f) The use of Lemma [7, Lm. 4.1.3], i.e. the identity $a^! \mathbb{Q}_X = \mathbb{Q}_Z[-2]$.

Issue (a) is resolved by taking the compactifications in §2.2.

Issue (b) is resolved by the forthcoming standard Lemma 5.1. We note that this issue (b) can also be resolved if the positive characteristic p of the ground field is bigger than the rank r of the Higgs bundles we are taking: then the stabilizers we deal with are i-roots of unity for $i = 1, \dots, r$, and they are thus reduced (finite cyclic).

Issue (c) is resolved in the forthcoming Lemma 5.2.

Issue (d) is resolved by [2, Thm. 20.4], where the authors prove a relative version of the Luna Slice Theorem. In particular, [2] implies that for a smooth affine B-scheme X with a free action of a smooth affine reductive group scheme G over B, if the GIT quotient X/G exists, then etale locally over B, X is etale locally isomorphic to the product of G and a B-scheme W, which, morally, is a slice transversal to the G-orbit.

Issue (e) is resolved by the forthcoming Lemma 5.3.

Issue (f) is resolved by O. Gabber Purity result [24, Tm. 2.2].

Lemma 5.1. *Let X be a quasi-projective scheme over a noetherian base scheme B. Let G be a finite flat group scheme over B that acts on X. Then a uniform geometric quotient $q : X \to X/G$ exists, the morphism q is finite, and the quotient X/G is quasi-projective over B.*

Proof. For the statements without the quasi-projectivity of X/G, a proof is contained in [34, Thm. 4.16]. When B is the spectrum of a field, a proof can also be found in [31, Thm. 12.1]. See also [35, Rmk. 4.2].

The quasi-projectivity of X/G over B is proved by [35, Prop. 4.5.(B′)] $\square$

Lemma 5.1 implies the following Lemma 5.2, which is needed in the proof of Lemma 5.3.

Lemma 5.2. *Let X/B be as in Lemma 5.1. Let H be a group scheme over B. Suppose that X/B admits an H-action, so that the uniform geometric quotient $q : X \to X/H$ exists. Let G be a finite flat closed group subscheme of H so that the quotient group scheme H/G exists and is reductive. Then we have a factorizaiton*

$$q : X \xrightarrow{q_1} X/G \xrightarrow{q_2} X/H,$$

where both q_1 and q_2 are uniform geometric quotients.

Proof. We first show that H/G acts on X/G: We have the short exact sequence

$$0 \to \mathcal{O}_X \otimes \mathcal{O}_{H/G} \to \mathcal{O}_X \otimes O_H \to \mathcal{O}_X \otimes \mathcal{O}_G \to 0.$$

Since the composition $\mathcal{O}_{X/G} \hookrightarrow \mathcal{O}_X \to \mathcal{O}_X \otimes \mathcal{O}_H \twoheadrightarrow \mathcal{O}_X \otimes \mathcal{O}_G$, where the middle morphism is the action comorphism, is trivial, we have a natural morphism $\mathcal{O}_{X/G} \to \mathcal{O}_X \otimes \mathcal{O}_{H/G}$, which factors through $\mathcal{O}_{X/G} \to \mathcal{O}_{X/G} \otimes \mathcal{O}_{H/G}$. One can check that this defines the comorphism of an H/G action on X/G.

By Lemma 5.1, we have that the uniform geometric quotient $q_1 : X \to X/G$ exists. Using [37, Thm. 3, Rmk. 8,10] as in the proof of Theorem 2.6.(2) in §3.4, we have that $q_2 : X/G \to (X/G)/(H/G)$ exists as a uniform categorical quotient, and that q_2 is a uniform geometric quotient if for any geometric point a of X/G, the set-theoretic image $\mu_a(H/G)$ of the orbit morphism μ_a is closed. The closedness of $\mu_a(H/G)$ is shown by the proof of [4, Lm. 3.3.1.(1)], thus q_2 is a uniform geometric quotient. The factorization $q = q_1 \circ q_2$ follows from the uniqueness of a geometric quotient. □

Recall that for any base scheme B and any abstract group G, there exists a unique constant group scheme over B associated with G, see [41, 03YW].

Lemma 5.3. *Assume the setup as in Lemma 5.1. Assume either one of the following additional assumptions on B or G:*

(i) G is the constant group scheme associated to an abstract finite group;
(ii) $B = k$ is an algebraically closed field;
(iii) B is of equal characteristics and G is the group scheme μ_N of N-th roots of unity for some $N \in \mathbb{Z}_{>0}$.

Then $(\overline{\mathbb{Q}}_\ell)_{X/G}$ is a direct summand of $q_*(\overline{\mathbb{Q}}_\ell)_X$.

Proof. Let us start with the case with assumption (i).

Let $q : X/ \to X/G$ be the uniform geometric quotient as in Lemma 5.1. We have that $(q_*(\overline{\mathbb{Q}}_\ell)_X)^G \cong (\overline{\mathbb{Q}}_\ell)_{X/G}$. Since $char(\overline{\mathbb{Q}}_\ell) = 0$, there exists the trace morphism

$$(s \mapsto \frac{1}{|G|}\sum_{g\in G} g^* s) : q_*(\overline{\mathbb{Q}}_\ell)_X \to (q_*(\overline{\mathbb{Q}}_\ell)_X)^G,$$

which defines a splitting of the inclusion $(\overline{\mathbb{Q}}_\ell)_{X/G} \hookrightarrow q_*(\overline{\mathbb{Q}}_\ell)_X$. The case with assumption (i) is then proved.

We now consider the case with assumption (ii): Let

$$1 \to G_0 \to G \to \pi_0(G) \to 1$$

be the "connected-étale short exact sequence" as in [32, p.114], i.e. G_0 is the unique connected normal subgroup scheme of G so that $\pi_0(G) := G/G_0$ is etale over k. Recall that G is reduced if and only if G_0 is trivial.

By Lemmata 5.1 and 5.2, the uniform geometric quotients $q_1 : X \to X/G_0$, $q_2 :$ $(X/G_0)/\pi_0(G)$ and $q : X \to X/G$ exist, and we have a factorization

$$q : X \xrightarrow{q_1} X/G_0 \xrightarrow{q_2} (X/G_0)/\pi_0(G).$$

Since G_0 is connected and k is algebraically closed, by [41, 054N] we have that for any field extension $K \supset k$, $(G_0)_K$ is also connected. Since $|(G_0)_K|$ is discrete, we have that $|(G_0)_K|$ is a singleton. Since $(q_1)_K : X_K \to (X/G_0)_K = X_K/(G_0)_K$ is also a geometric quotient, we have that $(q_1)_K$ is injective. By [41, 0154], we have that q_1 is universally injective, and thus purely inseparable. Since q_1 is also finite and surjective by Lemma 5.1, we have that q_1 is a universal homeomorphism [41, 04DF]. Therefore we have that $q_{1,*}(\overline{\mathbb{Q}}_\ell)_X = (\overline{\mathbb{Q}}_\ell)_{X/G_0}$, see [33, Rmk. 2.3.17].

Therefore, to show that $(\overline{\mathbb{Q}}_\ell)_{X/G_0}$ is a direct summand of $q_*(\overline{\mathbb{Q}}_\ell)_X$, it suffices to show that $(\overline{\mathbb{Q}}_\ell)_{X/G_0}$ is a direct summand of $q_{2,*}(\overline{\mathbb{Q}}_\ell)_{X/G_0}$. Note that $\pi_0(G)$ is associated with an abstract finite group, see e.g. [25, §8.21]. We are then reduced to the case with assumption (i). The case with assumption (ii) is thus finished.

For the case with assumption (iii), [2, Lm. 19.7] shows that there exists a subgroup scheme G_0 of μ_N so that fiber by fiber over B, G_0 restricts to the identity component of μ_N, and that the quotient group scheme G_0/μ_N is etale over B. Therefore the short exact sequence

$$1 \to G_0 \to \mu_N \to \mu_N/G_0 \to 1,$$

fiber by fiber over B, restricts to the étale-connected sequence in the proof in case (ii) above. Moreover, using the fact that μ_N is cyclic and B is of equal characteristic, we see that μ_N/G_0 is isomorphic to the group scheme associated with the abstract finite group $(\mu_N)_b(\kappa(b))$ for some closed point b in B with residue field $\kappa(b)$. Therefore, similar argument as in case (ii) finishes the proof in case (iii). $\square$

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